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$\mathcal{N} = 8$ Superspace Constraints for Three-dimensional Gauge Theories

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Abstract: We present a systematic analysis of the $\mathcal{N} = 8$ superspace constraints in three space-time dimensions. The general coupling between vector and scalar supermultiplets is encoded in an $SO(8)$ tensor W_{AB} which is a function of the matter fields and subject to a set of algebraic and super-differential relations. We show how the conformal BLG model as well as three-dimensional Yang-Mills theory provide solutions to these constraints and can both be formulated in this universal framework.

Keywords: Supersymmetric gauge theory, Chern-Simons Theories, Superspaces.

Contents

1. Introduction

Highly supersymmetric three dimensional gauge theories received tremendous attention over the last two years, in particular conformally symmetric matter Chern-Simons gauge theories. The origin of this interest was triggered by the formulation of the BLG-model [1, 2], a non-trivially interacting $\mathcal{N} = 8$ supersymmetric matter Chern-Simons gauge theory. It is an example of the sought-after theories describing the low energy dynamics of $M2$-branes and the conformally invariant fixed point of $\mathcal{N} = 8$ SYM theory [3]. Since then highly supersymmetric Chern-Simons gauge theories have been studied as examples of the AdS_4/CFT_3-correspondence and as solvable idealizations of condensed matter systems at the conformal fixed point [4]. Progress has been made especially for $\mathcal{N} \leq 6$ supersymmetric models. However, the

$\mathcal{N}=8$ case, corresponding to $M2$-branes in maximally symmetric compactified M-theory, remains notoriously intractable. The unitary BLG model is essentially unique with gauge group $SO(4)$ and arbitrary Chern-Simons level, whereas the $\mathcal{N}=6$ supersymmetric $U(N)\times U(N)$ ABJM model [5] has a proposed enhanced $\mathcal{N}=8$ supersymmetry for Chern-Simons levels $k=1,2$, but a manifest $\mathcal{N}=8$ supersymmetric formulation seems to be out of reach. It is generally accepted that these models are CFT's due to the quantized nature of the CS-coupling, for an explicit two-loop confirmation see [6]. For both kind of models Higgs mechanism have been introduced to study the flow to non-conformal SYM theories [5, 7].

Existing $\mathcal{N}=8$ superfield approaches [8, 9] using Nambu-brackets and pure spinors describe specifically the BLG model. In the work presented here we formulate and analyze the $\mathcal{N}=8$ superspace constraints for three-dimensional gauge theories which enables us to describe conformal Chern-Simons models and SYM theories on the same footing within a universal formalism. The matter sector is described by a real scalar superfield Φ^I transforming in the vector representation of the $SO(8)$ R-symmetry group. The gauge sector is described by a vector superfield which is an $SO(8)$ singlet. These superfields are subject to appropriate constraints to restrict the field content and we study the possible couplings of the gauge and mater superfields. The set of theories which are allowed by the consistency conditions of the constraints can be parametrized by an antisymmetric $SO(8)$ tensor W_{AB}, which is a function of the mater superfields subjected to the following concise $SO(8)$-projection conditions:

$$\nabla_{\alpha A} W_{BC}\Big|_{\mathbf{160_s}} = 0 \quad , \quad W_{IJ}\cdot\Phi_K\Big|_{\mathbf{160_v}} = 0 \quad ,$$

which will be explained in Detail in the main text. The $\mathcal{N}=8$ superspace formulation implemented here is necessarily on-shell, so that pure superspace geometrical considerations of the multiplet structure determine the dynamics of the system in terms of superfield equations of motions. Regarding this aspect this is in analogy with the approach of [1,2], where the closure of the susy algebra led to the component field e.o.m. However, given a manifest super-covariant formulation the consistency checks of [1, 2] are automatically incorporated and allow for a broader discussion of generalizations of the BLG-model.

We give two classes of solutions to the above conditions which describe BLG-type conformal Chern-Simons gauge theories and SYM theories. Lagrangian formulations are possible only in terms of component fields, and for the unitary BLG-model only for $SO(4)$ gauge group. The existence of a Lagrangian description at the conformal fixed point is not guaranteed, though favorable conditions of $\mathbb{Z}_k$ orbifold M-theory compactifications make the existence of the Lagrangian description by the ABJM models plausible [10], but there is a hitch, in the case of the proposed $\mathcal{N}=8$ supersymmetry with $k=1,2$ the theory is strongly coupled. Contrary to the four-dimensional $\mathcal{N}=4$ SYM theory there is no adjustable free parameter. In either case, existence of a strongly coupled Lagrangian or the lack of a Lagrangian description, quantum theoretical considerations have to be done by other means than perturbation theory within the models.

The superspace formulation that we present here provides a setting which allows the study of possible generalizations of BLG models and the determination of quantum corrections (to the e.o.m.) through symmetry considerations and by the rigidness of the $\mathcal{N} = 8$ superspace, circumventing perturbation theory. We give an outline of possible strategies in the end of this paper. The formulation of the dynamics in terms of superfield equations of motions carries enough information to investigate the modulispace of the theories as well as the possible chiral primary operators. Also the restrictions due the $\mathcal{N} = 8$ super conformal symmetry as discussed in superspace in [11] might be helpful for further investigations. A big challenge in the AdS_4/CFT_3 correspondence remains the understanding of the scaling of degrees of freedom with $N^{3/2}$ for the strongly coupled theory describing N $M2$-branes [12, 13].

Finally we want to mention a recent developement in $\mathcal{N} = 8$ light cone superspace [14, 15].

The paper is organized as follows. In section 2 we introduce the constraints for the free matter multiplet and the free Chern-Simons multiplet and the minimal coupling of the matter multiplet to it. In section 3 we deform the free Chern-Simons constraint and couple the matter sector to the gauge sector to obtain non-trivially interacting matter Chern-Simons theories and derive the above mentioned consistency conditions for the deformations. In section 4 we give particular solutions to the above conditions, leading to BLG models and SYM theories in their dual formulation. In section 5 we summarize our results and give an outlook on a number of future directions.

2. Free CS multiplet and minimally coupled matter

In this section we study the $\mathcal{N} = 8$ superspace description of the $\mathcal{N} = 8$ supermultiplet for free matter fields and matter fields minimally coupled to a free Chern-Simons multiplet, and thereby introduce the basic conventions and methods used in this paper. The $\mathcal{N} = 8$ superspace $\mathbb{R}^{2,1|16}$ is parametrized by coordinates $(x^{\alpha\beta}, \theta^{\alpha A})$, $A = 1, \dots, 8$, where the eight $\theta^{\alpha A}$ are real (Majorana) spinors in the $\mathbf{8_s}$ of the $SO(8)$ R-symmetry group and $x^{\alpha\beta}$ is a real symmetric matrix.[1] The susy covariant derivatives and the susy generators are given by the hermitian operators

$$D_{\alpha A} = \partial_{\alpha A} + i\theta^{\beta}_{A}\partial_{\alpha\beta} \ , \qquad Q_{\alpha A} = \partial_{\alpha A} - i\theta^{\beta}_{A}\partial_{\alpha\beta} \ , \tag{2.1}$$

such that $\{D_{\alpha A}, Q_{\beta B}\} = 0$ and

$$\{Q_{\alpha A}, Q_{\beta B}\} = -\{D_{\alpha A}, D_{\beta B}\} = -2i\delta_{AB}\partial_{\alpha\beta} \ . \tag{2.2}$$

[1] For more details regarding the notation see the appendix.

$SO(8)$ indices are raised/lowered with a Kronecker-delta and thus one does not have to pay special attention to their position. We will also use gauge covariant derivatives in superspace, which we introduce as follows:

$$\nabla_{\alpha\beta} = \partial_{\alpha\beta} + \mathcal{A}_{\alpha\beta} \quad \text{and} \quad \nabla_{\alpha A} = D_{\alpha A} + \mathcal{A}_{\alpha A} \ . \tag{2.3}$$

When acting in complex bundles the physicality condition would be that the bosonic superspace connection $\mathcal{A}_{\alpha\beta}$ is anti-hermitian, while the fermionic one, $\mathcal{A}_{\alpha A}$, is hermitian, but we consider here real bundles and therefore the property under complex conjugation is the primary issue. To have the same conjugation property as for the differential operators we require that the bosonic superspace connection $\mathcal{A}_{\alpha\beta}$ is real, while the fermionic one, $\mathcal{A}_{\alpha A}$, is imaginary. Both connections carry a representation of the gauge symmetry structure group and complex conjugation has to be defined accordingly. This and the action of the covariant derivatives on different fields will be discussed in detail when considering specific models.

2.1 The free matter multiplet

Superfield constraints

The $\mathcal{N} = 8$ scalar multiplet consists of eight real scalars and eight Majorana-fermions $(\phi^I, \psi_{\alpha\dot{A}})$ in the $\mathbf{8_v}$ and $\mathbf{8_c}$, respectively, of $SO(8)$. The free field equations are given by

$$\Box\phi^I = 0 \ , \qquad \varepsilon^{\beta\gamma}\partial_{\alpha\beta}\psi_{\gamma\dot{A}} = 0 \ , \tag{2.4}$$

where $\Box := \partial^{\alpha\beta}\partial_{\alpha\beta}$. The fields ϕ^I and consequently $\psi_{\alpha\dot{A}}$ may carry an additional representation of some internal (global) symmetry group, which we do not indicate here but will be discussed in detail when we consider the interacting theories and systematically gauge these symmetries.

For finding superfields encoding this on-shell component multiplet it is therefore natural to start with a real scalar superfield Φ^I in the $\mathbf{8_v}$ of $SO(8)$ (and in the same representation of a possible internal symmetry as ϕ^I), and impose necessary constraints to appropriately restrict the component field content. At first order in $\theta^{\alpha A}$, this field contains components which transform as $\mathbf{8_v} \otimes \mathbf{8_s} = \mathbf{8_c} \oplus \mathbf{56_c}$ under $SO(8)$.[2] Comparing to the field content of the component multiplet, it follows that one has to eliminate the unwanted component field in the $\mathbf{56_c}$. In a susy covariant way this is achieved by imposing

$$D_{\alpha A}\Phi^I\big|_{\mathbf{56_c}} \stackrel{!}{=} 0 \qquad \Longleftrightarrow \qquad D_{\alpha A}\Phi^I = \tfrac{1}{8}(\Gamma^I\bar{\Gamma}^J)_{AB}D_{\alpha B}\Phi^J \ . \tag{2.5}$$

In [9] a pure spinor superfield formulation of the BLG model was given and the equivalent to (2.5) was found as an invariance condition for the pure spinor wave-function.

[2] For details of $SO(8)$ representations and various Γ-relations see the appendix B. Decompositions of tensor products of $SO(8)$ representations can be computed with the program LiE [16] or found in [17].

The constraint (2.5) implies the existence of a fermionic superfield $\Psi_{\alpha\dot{A}}$ such that $D_{\alpha A}\Phi^I$ is explicitly restricted to the $\mathbf{8_c}$:

$$D_{\alpha A}\Phi^I = i\Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}} \quad , \tag{2.6}$$

and for our purposes and in particular for applying the methods developed in [18,19] it will be more convenient to work with this form of the constraint. Equation (2.6) can be solved explicitly for $\Psi_{\alpha\dot{A}}$ which by inserting gives back (2.5). This form of the constraint resembles the form of the "super-embedding" equation of [8], where the BLG model was realized in terms of Nambu-brackets. The similarity will become more evident in the interacting case.

The fermionic superfield $\Psi_{\alpha\dot{A}}$ is not completely free, but is itself restricted due to the integrability condition of the constraint (2.6). With (2.2) this gives:

$$2\delta_{AB}\partial_{\alpha\beta}\Phi^I = \Gamma^I_{A\dot{A}}D_{\beta B}\Psi_{\alpha\dot{A}} + \Gamma^I_{B\dot{A}}D_{\alpha A}\Psi_{\beta\dot{A}} \ , \tag{2.7}$$

which allows only the $(\mathbf{3},\mathbf{8_v})$ part of $D_{\alpha A}\Psi_{\beta\dot{A}}$ to be nonzero, where the first entry refers to the $so(2,1)$ representation. We demonstrate here for once the procedure how we resolve such equations systematically. Decomposing $D_{\alpha A}\Psi_{\beta\dot{A}}$ according to its irreducible representations

$$\underbrace{D_{\alpha A}\Psi_{\beta\dot{A}}}_{(\mathbf{2}\otimes\mathbf{2},\mathbf{8_s}\otimes\mathbf{8_c})} = \Gamma^I_{A\dot{A}}\underbrace{(\varepsilon_{\alpha\beta}a^I + a^I_{\alpha\beta})}_{(\mathbf{1}\oplus\mathbf{3},\mathbf{8_v})} + \Gamma^{IJK}_{A\dot{A}}\underbrace{(\varepsilon_{\alpha\beta}b_{IJK} + b_{\alpha\beta IJK})}_{(\mathbf{1}\oplus\mathbf{3},\mathbf{56_v})} \ , \tag{2.8}$$

where the 3-form[3] $b_{IJK} = b_{[IJK]}$ is the $\mathbf{56_v}$ and so is the $so(2,1)$ vector $b_{\alpha\beta IJK}$. Inserting this decomposition into (2.7) shows that only the $(\mathbf{3},\mathbf{8_v})$ part $a^I_{\alpha\beta}$ can be non-zero and is given by the l.h.s. The integrability condition (2.7) then implies

$$D_{\alpha A}\Psi_{\beta\dot{A}} = \Gamma^I_{A\dot{A}}\partial_{\alpha\beta}\Phi^I \ . \tag{2.9}$$

The constraint (2.6) and its integrability condition (2.9) are the primary relations/conditions from which we derive all further consequences. From now on we will often refer to the constraint and its integrability condition as just the "(superfield) constraints". Using (2.2) to express x-space derivatives in terms of superderivatives one obtains that the superfields Φ^I, $\Psi_{\alpha\dot{A}}$ subject to the constraints (2.6), (2.9) satisfy the free superfield e.o.m

$$\varepsilon^{\beta\gamma}\partial_{\alpha\beta}\Psi_{\gamma\dot{A}} = 0 \ , \qquad \Box\Phi^I = 0 \ , \tag{2.10}$$

where $\Box := \partial^{\alpha\beta}\partial_{\alpha\beta}$. Thus the full superfields and therefore their lowest components $\phi^I := \Phi^I|_{\theta=0}$, $\psi_{\alpha\dot{A}} := \Psi_{\alpha\dot{A}}|_{\theta=0}$ (which are nonzero as we will see), satisfy the free e.o.m. (2.4), as desired. One could expect to get an additional condition from the integrability condition of (2.9) but it is easy to see that it reduces to the superfield equations of motion (2.10).

[3]The explicit form of a tensor in a representation of given dimension and the symmetries of these tensors are conveniently obtained via Young diagrams, see for example [20], though their applicability is restricted for (special) orthogonal groups.

Superfield expansion

Following [18,19] we now derive recursion relations which determine the θ-expansion of the superfields. Defining the homogeneity operator

$$\mathcal{R} := \theta^{\alpha A} D_{\alpha A} = \theta^{\alpha A}\partial_{\alpha A} \ , \tag{2.11}$$

which satisfies $\mathcal{R}(\theta^{\alpha_1 A_1}\ldots\theta^{\alpha_n A_n}) = n\ \theta^{\alpha_1 A_1}\ldots\theta^{\alpha_n A_n}$, one obtains by contracting the constraints (2.6), (2.9) with $\theta^{\alpha A}$ the *recursion relations*

$$\begin{aligned} \mathcal{R}\Phi^I &= i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}} \ , \\ \mathcal{R}\Psi_{\beta\dot{A}} &= \theta^{\alpha A}\Gamma^I_{A\dot{A}}\partial_{\alpha\beta}\Phi^I \ , \end{aligned} \tag{2.12}$$

which due to the property of $\mathcal{R}$ give the $(n+1)$'th order in θ of the superfields on the l.h.s in terms of the n'th order of the superfields on the r.h.s.

The recursions (2.12) determine the complete superfield expansion in terms of the lowest components ϕ^I and $\psi_{\alpha\dot{A}}$, but without any further conditions on them and thus represent the non-dynamical part of the constraint equations. The resulting superfield expansion is:

$$\begin{aligned} \Phi^I &= \phi^I + i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\psi_{\alpha\dot{A}} + \tfrac{i}{2}\theta^{\alpha A}\theta^{\beta B}\Gamma^{IJ}_{AB}\partial_{\alpha\beta}\phi^J + \ldots \ , \\ \Psi_{\beta\dot{A}} &= \psi_{\beta\dot{A}} + \theta^{\alpha A}\Gamma^I_{A\dot{A}}\partial_{\alpha\beta}\phi^I + \tfrac{i}{2}\theta^{\alpha A}\theta^{\gamma B}\Gamma^I_{A\dot{A}}\Gamma^I_{B\dot{B}}\partial_{\alpha\beta}\psi_{\gamma\dot{B}} + \ldots \ . \end{aligned} \tag{2.13}$$

Given that the supersymmetry variation of a superfield F is $\delta F = \epsilon^{\alpha A}Q_{\alpha A}F$ one obtains from (2.1) the following transformations for the component fields:

$$\delta\phi^I = i\epsilon^{\alpha A}\Gamma^I_{A\dot{A}}\psi_{\alpha\dot{A}} \ , \qquad \delta\psi_{\beta\dot{A}} = \epsilon^{\alpha A}\Gamma^I_{A\dot{A}}\partial_{\alpha\beta}\phi^I \ , \tag{2.14}$$

which by construction are symmetries of the e.o.m. (2.4).

Concluding, we have shown that the superfield constraints (2.6), (2.9) imply a superfield expansion exclusively in terms of the component multiplet $(\phi^I, \psi_{\alpha\dot{A}})$ with the supersymmetry transformations (2.14). Moreover, these superfields satisfy the free superfield e.o.m. (2.10) and thus the component fields satisfy the free e.o.m. (2.4). In the rest of this section, we will show that vice versa the on-shell component fields define superfields which satisfy the constraints (2.6), (2.9) so that these two descriptions are completely equivalent. In particular, the constraints (2.6), (2.9) do not imply any further restrictions on the components.

Equivalence to component e.o.m.

We now start from the on-shell component multiplet $(\phi^I, \psi_{\alpha\dot{A}})$, which is assumed to satisfy the free e.o.m (2.4), which are supersymmetric under the transformations (2.14), and show that this defines superfields satisfying the constraints (2.6), (2.9).

Susy covariance. We use the recursion relations (2.12) to define superfields out of the component multiplet $(\phi^I, \psi_{\alpha\dot{A}})$. For the first few terms in the θ-expansion

(2.13) we have already shown that the component supersymmetry transformations (2.14) can be written as $\delta\Phi^I = \epsilon Q\Phi^I$, $\delta\Psi_{\alpha\dot{A}} = \epsilon Q\Psi_{\alpha\dot{A}}$, with $Q_{\alpha A}$ given in (2.1). The recursion relations (2.12) are not susy covariant and one has to check explicitly if they define a consistent superfield[4], i.e. that susy transformed superfields satisfy the same recursion relations.

Acting with $\epsilon^{\alpha A}Q_{\alpha A}$ on the recursion relations (2.12) one obtains

$$\begin{aligned} \mathcal{R}\delta\Phi^I &= i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\delta\Psi_{\alpha\dot{A}} - \epsilon^{\alpha A}[D_{\alpha A}\Phi^I - i\Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}}] \ , \\ \mathcal{R}\delta\Psi_{\beta\dot{A}} &= \theta^{\alpha A}\Gamma^I_{A\dot{A}}\partial_{\alpha\beta}\delta\Phi^I - \epsilon^{\alpha A}[D_{\alpha A}\Psi_{\beta\dot{A}} - \Gamma^I_{A\dot{A}}\partial_{\alpha\beta}\Phi^I] \ . \end{aligned} \tag{2.15}$$

Thus the susy variations satisfy the same recursions as the original fields *iff* the superfield constraints (2.6), (2.9) are satisfied. To show that the component e.o.m. imply these constraints we first prove that they imply the full superfield e.o.m.

Superfield e.o.m. To zeroth order in θ, the superfields equal the components $(\phi^I, \psi_{\alpha\dot{A}})$ and thus per construction satisfy the e.o.m. To show that this implies that they are satisfied in all orders in θ we derive a recursive system for the superfield e.o.m.,

$$\mathcal{E}_{\alpha\dot{A}} := \varepsilon^{\beta\gamma}\partial_{\alpha\beta}\Psi_{\gamma\dot{A}} \quad , \qquad \mathcal{E}^I := \Box\Phi^I \quad . \tag{2.16}$$

Using exclusively the recursion relations (2.12) one obtains[5]

$$\begin{aligned} \mathcal{R}\mathcal{E}_{\alpha\dot{A}} &= -\theta^{\beta A}\Gamma^I_{A\dot{A}}\varepsilon_{\beta\alpha}\mathcal{E}^I \quad , \\ \mathcal{R}\mathcal{E}^I &= -i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\varepsilon^{\beta\gamma}\nabla_{\alpha\beta}\mathcal{E}_{\gamma\dot{A}} \quad . \end{aligned} \tag{2.17}$$

As to lowest order the e.o.m. are satisfied, i.e. $\mathcal{E}_{\alpha\dot{A}}|_{\theta=0} = \mathcal{E}^I|_{\theta=0} = 0$, these recursions imply that $\mathcal{E}_{\alpha\dot{A}}$, $\mathcal{E}^I$ vanish to all orders. Thus the component e.o.m. (2.4) imply the superfield e.o.m. (2.10) for the superfields defined by (2.12).

Constraints. In the last step we show that the superfield e.o.m. (2.10) imply the constraint equations (2.6), (2.9). To this end we introduce the abbreviations

$$\begin{aligned} \mathcal{C}^I_{\alpha A} &= D_{\alpha A}\Phi^I - i\ \Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}} \quad , \\ \mathcal{C}_{\alpha\beta A\dot{A}} &= D_{\alpha A}\Psi_{\beta\dot{A}} - \Gamma^I_{A\dot{A}}\partial_{\alpha\beta}\Phi^I \quad . \end{aligned} \tag{2.18}$$

Using the recursion relation (2.12) one obtains the following recursions for the constraints $\mathcal{C}^I_{\alpha A}$ and $\mathcal{C}_{\alpha\beta A\dot{A}}$:

$$\begin{aligned} (1+\mathcal{R})\,\mathcal{C}^I_{\alpha A} &= i\theta^{\beta B}\Gamma^I_{B\dot{A}}\mathcal{C}_{\alpha\beta A\dot{A}} \quad , \\ (1+\mathcal{R})\,\mathcal{C}_{\alpha\beta A\dot{A}} &= -\theta^{\gamma B}\Gamma^I_{B\dot{A}}\partial_{\beta\gamma}\mathcal{C}^I_{\alpha A} \quad , \end{aligned} \tag{2.19}$$

[4]This is a complementary approach to find the correct superfield constraints for a given multiplet with susy transformations (2.14) which upon comparing with (2.12) define the recursion relations so that the superfield expansion is generated by consecutive susy transformations.

[5]Here and on many other occasions we use the fact that the total antisymmetrization of three spinor indices, which take two values, gives zero.

where in the second relation we used the fermionic superfield e.o.m. (2.10). These recursions imply that the constraints $\mathcal{C}^I_{\alpha A}$ and $\mathcal{C}_{\alpha\beta A\dot{A}}$ vanish in all orders in θ. We thus have proved that the on-shell multiplet $(\phi^I, \psi_{\alpha\dot{A}})$ with equations of motion (2.4) is completely equivalent to the superfields $(\Phi^I, \Psi_{\alpha\dot{A}})$ satisfying the constraints (2.6), (2.9).

2.2 Free Chern-Simons multiplet

In general, we will be interested in theories whose matter content is given by a number of scalar super-multiplets. At the linearized level, such theories are described by N superfields $\Phi^{Ia}, \Psi^a_{\alpha\dot{A}}$, subject to the constraint (2.6), where the additional index $a = 1, \dots N$, labels the different super-multiplets. The obvious global symmetry group (besides the $SO(8)$ R-symmetry, which we will not gauge) of the system is $GL(N,\mathbb{R}) \ltimes \mathbb{T}(8N)$ acting as

$$\delta\Phi^I = \Lambda \cdot \Phi^I + C^I \ , \qquad \delta\Psi_{\alpha\dot{A}} = \Lambda \cdot \Psi_{\alpha\dot{A}} \ , \tag{2.20}$$

with a matrix $\Lambda \in \mathfrak{gl}(N,\mathbb{R})$ (where we have suppressed the explicit indices a), which are obviously symmetries of (2.6). The shifts $\mathbb{T}(8N)$ act exclusively on the scalars Φ^I.[6]

In the interacting theories, a subset of these symmetries will be gauged by selecting a subalgebra $\mathfrak{g}$

$$\begin{aligned} \langle T_M \rangle &= \mathfrak{g} \subset \mathfrak{gl}(N,\mathbb{R}) \oplus_s \mathfrak{t}(8N) \ , \\ [T_M, T_N] &= f_{MN}{}^K \, T_K \ , \end{aligned} \tag{2.21}$$

spanned by generators T_M . Choosing $\mathfrak{g}$ to have non-trivial intersection with $\mathfrak{t}(8N)$ a priori breaks the $SO(8)$ R-symmetry. The corresponding gauge superfields appearing in the covariant derivatives (2.3) are thus given by

$$\mathcal{A}_{\alpha A} = \mathcal{A}^M_{\alpha A} i T_M \ , \qquad \mathcal{A}_{\alpha\beta} = \mathcal{A}^M_{\alpha\beta} T_M \ . \tag{2.22}$$

Assuming a real representations for the generators T_M, this gives the right conjugation property for real $\mathcal{A}^M_{\alpha A}$ and $\mathcal{A}^M_{\alpha\beta}$, as defined below (2.3).

Note that at this stage we do not encounter three algebras as introduced in [1,2,21]. We will see in later sections how the defining relation of these three algebras, the *fundamental identity* for a rank four tensor, is a natural consequence for conformal models based on Lie algebras.

[6]The component field equations (2.4) would allow also for global shifts $\delta\psi_{\alpha\dot{A}} = \zeta_{\alpha\dot{A}}$ of the fermionic component field and thus of the superfield $\Psi_{\alpha\dot{A}}$. In view of the superfield expansion (2.13) this would imply a corresponding θ-dependent shifts $\delta\Phi^I = C^I + i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\zeta_{\alpha\dot{A}}$ in the bosonic superfield Φ^I and represent a more involved symmetry of the constraints (2.6), (2.9). We do not consider this possibility here.

Introducing the gauge parameter field $\Omega = \Omega^M T_M$, the local versions of (2.20) and the gauge transformations of the gauge fields can be compactly written as

$$\begin{aligned} \delta\Phi^I &= \Omega \cdot \Phi^I \ , \quad \delta\Psi_{\alpha\dot{A}} = \Omega \cdot \Psi_{\alpha\dot{A}} \ , \\ \delta\mathcal{A}_{\alpha A} &= -\nabla_{\alpha A}\Omega \ , \quad \delta\mathcal{A}_{\alpha\beta} = -\nabla_{\alpha\beta}\Omega \ , \end{aligned} \tag{2.23}$$

where the gauge fields transform in the adjoint of (2.21) and the matter superfields now transform in some representation of the gauge algebra which is indicated by the dot.

The field strengths are given in the usual way through (anti)commutators of the connections minus torsion terms, i.e.

$$\begin{aligned} \mathcal{F}_{\alpha A,\beta B} &= \{\nabla_{\alpha A}, \nabla_{\beta B}\} - 2i\delta_{AB}\nabla_{\alpha\beta} \ , \\ \mathcal{F}_{\alpha\beta,\gamma\delta} &= [\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] \ , \\ \mathcal{F}_{\alpha\beta,\gamma C} &= [\nabla_{\alpha\beta}, \nabla_{\gamma C}] \ . \end{aligned} \tag{2.24}$$

Free CS superfield constraints

The gauge superfields ($\mathcal{A}_{\alpha A}$, $\mathcal{A}_{\alpha\beta}$) contain way to many component fields and one has to impose constraints to obtain a physically meaningful multiplet. In particular, $\mathcal{A}_{\alpha A}$ contains a second component vector field with the same gauge-transformation as the lowest component of $\mathcal{A}_{\alpha\beta}$. It has turned out to be promising to impose (partial) flatness conditions on the bi-spinor field strength, here $\mathcal{F}_{\alpha A,\beta B}$, to eliminate unphysical degrees of freedom [22–25]. In many cases this corresponds to an underlying geometric structure of twistors and pure spinors [24–27]. In particular, the unwanted vector field is eliminated by the "conventional constraint", see for example [28, 29], where a symmetrized part of $\mathcal{F}_{\alpha A,\beta B}$ is set to zero. Since we are interested here in the free multiplet we impose a constraint which is rather strong in three dimensions and require the entire $\mathcal{F}_{\alpha A,\beta B}$ to vanishes. Relaxations of this constraint will be discussed when we consider non-minimally interacting theories. Thus, for this section we set

$$\mathcal{F}_{\alpha A,\beta B} \overset{!}{=} 0 \quad \Longleftrightarrow \quad \{\nabla_{\alpha A}, \nabla_{\beta B}\} = 2i\delta_{AB}\nabla_{\alpha\beta} \ . \tag{2.25}$$

As in the case of the matter superfield constraint (2.6) the right r.h.s. of (2.25) is not completely free but has to satisfy certain conditions so that it factorizes into an anti-commutator. The analogon to the integrability condition (2.9) are the Bianchi identities, which are simply obtained from the super-Jacobi identities for the covariant derivatives:[7]

$$\begin{aligned} &\sum_{\text{cyclic}} [\nabla_{\alpha A}, \{\nabla_{\beta B}, \nabla_{\gamma C}\}] \equiv 0 \ , \qquad \sum_{\text{cyclic}} (-1)^\pi \{\nabla_{\alpha A}, [\nabla_{\beta B}, \nabla_{\gamma\delta}]\} \equiv 0 \ , \\ &\sum_{\text{cyclic}} [\nabla_{\rho A}, [\nabla_{\alpha\beta}, \nabla_{\gamma\delta}]] \equiv 0 \quad , \qquad \sum_{\text{cyclic}} [\nabla_{\alpha\beta}, [\nabla_{\gamma\delta}, \nabla_{\rho\sigma}]] \equiv 0 \ . \end{aligned} \tag{2.26}$$

[7]The exponent π in the second identity counts the cyclic permutations where (anti)commutators are distributed correspondingly to the occurrence of bosonic/fermionic connections

First of all, these identities imply nontrivial conditions in case (2.25) appears, i.e. for the first and second identity with three fermionic and two fermionic covariant derivatives, respectively. In these cases one obtains:

$$
\begin{aligned}
\delta_{AB}\mathcal{F}_{\alpha\beta,\gamma C} + \delta_{AC}\mathcal{F}_{\alpha\gamma,\beta B} + \delta_{BC}\mathcal{F}_{\beta\gamma,\alpha A} &= 0 \ , \\
\nabla_{\alpha A}\mathcal{F}_{\gamma\delta,\beta B} + \nabla_{\beta B}\mathcal{F}_{\gamma\delta,\alpha A} &= 2i\delta_{AB}\mathcal{F}_{\gamma\delta,\alpha\beta} \ .
\end{aligned} \tag{2.27}
$$

Decomposing the two equations analogously to (2.8) into irreducible representations of $so(2,1)$ and $SO(8)$, one finds that the two Bianchi identities imply that also the other two components of the super field strength vanish, i.e.

$$
\mathcal{F}_{\alpha\beta,\gamma C} = 0 = \mathcal{F}_{\alpha\beta,\gamma\delta} \ . \tag{2.28}
$$

With these strong equations for the commutators/field strengths the other two Bianchi identities in (2.26) are identically fulfilled and do not impose further conditions. The second equation in (2.28), which follows with the help of the first one, is the free Chern-Simons superfield equation of motion. To see what this implies at the level of component fields we again follow the strategy of [18, 19] to obtain the superfield expansion.

Superfield expansion

To eliminate the gauge degrees of freedom in the gauge superfields and to be able to apply the same recursive method as in (2.12) one imposes the "transverse" gauge [18] on the fermionic gauge superfields,

$$
\theta^{\alpha A}\mathcal{A}_{\alpha A} = 0 \quad \Longrightarrow \quad \mathcal{R} = \theta^{\alpha A}\nabla_{\alpha A} \ . \tag{2.29}
$$

This fixes the gauge freedom (2.23) up to pure x-space dependent gauge transformations and is thus a kind of WZ-gauge. Moreover, it allows to write the recursion operator $\mathcal{R}$ (2.11) in a covariant form. Therefore, contracting the constraint (2.25) and the first Bianchi identity (2.28) with $\theta^{\gamma C}$ one obtains the recursion relations

$$
\begin{aligned}
(1+\mathcal{R})\ \mathcal{A}_{\beta B} &= 2i\theta^{\alpha}_{B}\mathcal{A}_{\alpha\beta} \ , \\
\mathcal{R}\ \mathcal{A}_{\alpha\beta} &= 0 \ .
\end{aligned} \tag{2.30}
$$

This gives the rather trivial superfield expansions,

$$
\mathcal{A}_{\alpha A} = i\theta^{\beta}_{A}A_{\alpha\beta} \ , \qquad \mathcal{A}_{\alpha\beta} = A_{\alpha\beta} \ , \tag{2.31}
$$

where the lowest component $A_{\alpha\beta} := \mathcal{A}_{\alpha\beta}|_{\theta=0}$ is the vector field in x-space. The condition due to the second Bianchi identity in (2.28) thus implies the component field equations

$$
F_{\alpha\beta,\gamma\delta} = 0 \ , \tag{2.32}
$$

which is the free Chern-Simons e.o.m. Consequently, the multiple associated with the constraint (2.25) contains a single component field, the vector field $A_{\alpha\beta}$, which is pure gauge and therefore has no dynamical degrees of freedom.

Equivalence to component e.o.m.

To prove that a component vector field $A_{\alpha\beta}$, satisfying (2.32) is equivalent to the full constraint (2.25) is trivial in this case. Adopting the superfield expansions (2.31) one immediately sees that these superfields satisfy the constraint (2.25) and the Bianchi identities (2.28) due to the component field e.o.m. (2.32). Nevertheless, we consider the susy-covariance of the recursion relations and the susy-transformations of the component field. Defining the superfield transformations as before, i.e. $\delta\mathcal{A}_{\alpha A} = \epsilon Q\mathcal{A}_{\alpha A}$ and $\delta\mathcal{A}_{\alpha\beta} = \epsilon Q\mathcal{A}_{\alpha\beta}$, and acting with ϵQ on the recursions (2.30) one finds:

$$
\begin{aligned}
(1+\mathcal{R})\,\delta\mathcal{A}_{\alpha A} &= 2i\theta^\beta_B\delta\mathcal{A}_{\alpha\beta} - \epsilon^{\beta B}\mathcal{F}_{\alpha A,\beta B} - \nabla_{\alpha A}\Lambda\ , \\
\mathcal{R}\delta\mathcal{A}_{\alpha\beta} &= 0 + \epsilon^{\gamma C}\mathcal{F}_{\alpha\beta,\gamma C} - \nabla_{\alpha\beta}\Lambda\ ,
\end{aligned}
\tag{2.33}
$$

where in both cases the last term is a field dependent supergauge transformation with the gauge parameter field

$$
\Lambda = \epsilon^{\alpha A}\mathcal{A}_{\alpha A} = i\epsilon^{\alpha A}\theta^\beta_A A_{\alpha\beta}\ . \tag{2.34}
$$

Thus up to the constraint (2.25) and the first Bianchi identity in (2.28) the recursion relations are susy covariant modulo field dependent gauge transformations. The occurrence of the field dependent gauge transformation is not surprising since the "transverse" gauge (2.29) is not susy covariant. In the same fashion, using the (component) field e.o.m. (2.32) and in view of the superfield expansion (2.31) one obtains for the supersymmetry transformations

$$
\begin{aligned}
\delta\mathcal{A}_{\alpha A} &= \epsilon^{\gamma C}Q_{\gamma C}\mathcal{A}_{\alpha A} =: i\theta^\beta_A\ \delta A_{\alpha\beta} = -\nabla_{\alpha A}\Lambda\ , \\
\delta\mathcal{A}_{\alpha\beta} &= \epsilon^{\gamma C}Q_{\gamma C}\mathcal{A}_{\alpha\beta} =:\ \delta A_{\alpha\beta}\ = -\nabla_{\alpha\beta}\Lambda\ .
\end{aligned}
\tag{2.35}
$$

Thus the susy transformations of the superfields are (on-shell) pure gauge transformations with the field dependent parameter Λ (2.34). These gauge transformations do not have a component in the appropriate order of θ such that the supersymmetry transformation of the component field in (2.35) is just

$$
\delta A_{\alpha\beta} = 0\ . \tag{2.36}
$$

The curious fact for the free Chern-Simons case, that a multiplet with a single component field, $A_{\alpha\beta}$, is nevertheless (on-shell) supersymmetric was discussed in [3]. Here we obtain the same result in a super-covariant way.

2.3 Minimal Coupling

We now covariantize the procedure of section 2.1 by minimally coupling the matter superfields to gauge superfields subject to the constraint (2.25). Given that the gauge field remains pure gauge this seems to be trivial, but it sets the formalism for the next section, where we consider non-linear deformations which lead to a non-trivially coupled system.

Superfield Constraints

The covariantized constraint (2.6) with minimal coupling is

$$\nabla_{\alpha A}\Phi^I = i\Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}} \ . \tag{2.37}$$

Using the gauge field constraint (2.25) and the Bianchi identities (2.28), the integrability condition of (2.37) reduces to

$$\nabla_{\alpha A}\Psi_{\beta\dot{A}} = \Gamma^I_{A\dot{A}}\nabla_{\alpha\beta}\Phi^I \ . \tag{2.38}$$

Further, using the gauge field constraint (2.25) to express $\nabla_{\alpha\beta}$ in terms of superderivatives and the Bianchi identities (2.28) together with (2.37), (2.38), the superfield e.o.m. compute to

$$\varepsilon^{\beta\gamma}\nabla_{\alpha\beta}\Psi_{\gamma\dot{A}} = 0\,, \qquad \nabla^2\Phi^I = 0 \ , \tag{2.39}$$

where $\nabla^2 = \nabla^{\alpha\beta}\nabla_{\alpha\beta}$.

Superfield expansion

To obtain the superfield expansion we again impose the "transverse" gauge (2.29). Contracting the constraints (2.37), (2.38) with $\theta^{\alpha A}$ one obtains the recursion relations

$$\begin{aligned} \mathcal{R}\Phi^I &= i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}} \ , \\ \mathcal{R}\Psi_{\beta\dot{A}} &= \theta^{\alpha A}\Gamma^I_{A\dot{A}}\nabla_{\alpha\beta}\Phi^I \ . \end{aligned} \tag{2.40}$$

This again defines the superfield expansion in terms of the lowest components $\phi^I = \Phi^I|_{\theta=0}$ and $\psi_{\alpha\dot{A}} = \Psi_{\alpha\dot{A}}|_{\theta=0}$, where things considerable simplify due to the fact that for the free Chern-Simons multiplet in the "transverse" gauge (2.31) one has

$$\nabla_{\alpha\beta} = \overset{\circ}{\nabla}_{\alpha\beta} := \partial_{\alpha\beta} + A_{\alpha\beta} \ , \tag{2.41}$$

i.e. only the lowest component of the vector superfield is present in the superconnection $\nabla_{\alpha\beta}$.[8] Hence the superfield expansion is given by

$$\begin{aligned} \Phi^I &= \phi^I + i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\psi_{\alpha\dot{A}} + \tfrac{i}{2}\theta^{\alpha A}\theta^{\beta B}\Gamma^{IJ}_{AB}\overset{\circ}{\nabla}_{\alpha\beta}\phi^J + \dots \ , \\ \Psi_{\beta\dot{A}} &= \psi_{\beta\dot{A}} + \theta^{\alpha A}\Gamma^I_{A\dot{A}}\overset{\circ}{\nabla}_{\alpha\beta}\phi^I + \tfrac{i}{2}\theta^{\alpha A}\theta^{\gamma B}\Gamma^I_{A\dot{A}}\Gamma^I_{B\dot{B}}\overset{\circ}{\nabla}_{\alpha\beta}\psi_{\gamma\dot{B}} + \dots, \end{aligned} \tag{2.42}$$

and therefore the lowest components of the superfield e.o.m. (2.39) imply the corresponding e.o.m for the component fields ϕ^I and $\psi_{\alpha\dot{A}}$. We have thus shown that the constraints and integrability conditions/Bianchi identities (2.25), (2.28), (2.37),

[8] By '$\circ$' we generically denote the lowest component of a superfield: $\overset{\circ}{\Phi} := \Phi|_{\theta=0} = \phi$, etc.

(2.38) give a minimally coupled Chern-Simons multiplet $(A_{\alpha\beta}, \phi^I, \psi_{\alpha\dot{A}})$ with the e.o.m.

$$F_{\alpha\beta,\gamma\delta} = 0 \;, \qquad \overset{\circ}{\nabla}{}^2\,\phi^I = 0 \;, \qquad \varepsilon^{\beta\gamma}\overset{\circ}{\nabla}_{\alpha\beta}\psi_{\gamma\dot{A}} = 0 \;. \tag{2.43}$$

The supersymmetry transformations of the matter multiplet are obtained from the superfield expansion (2.42) in the usual way,

$$\delta\Phi^I = \epsilon^{\alpha A}Q_{\alpha A}\Phi^I =: (\delta\phi^I + i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\delta\psi_{\alpha\dot{A}}\ldots) + \Lambda\cdot\Phi^I \;, \tag{2.44}$$

where as in the case of the gauge multiplet we obtain the component field transformations modulo a compensating gauge transformation with the same gauge parameter Λ (2.34). The resulting supersymmetry transformations are then

$$\begin{aligned} &\delta\phi^I = i\epsilon^{\alpha A}\Gamma^I_{A\dot{A}}\psi_{\alpha\dot{A}} \;, \;\; \delta\psi_{\alpha\dot{A}} = \epsilon^{\beta A}\Gamma^I_{A\dot{A}}\overset{\circ}{\nabla}_{\alpha\beta}\phi^I \;, \\ &\delta A_{\alpha\beta} = 0 \;, \end{aligned} \tag{2.45}$$

where for completeness we have rewritten the transformation of the gauge field (2.36). These supersymmetry transformations again resemble the recursion relations (2.40), (2.30) of the associated superfields.

Equivalence to component e.o.m.

We have already shown that the component vector field $A_{\alpha\beta}$ subject to the free Chern-Simons e.o.m. (2.32) is equivalent to the gauge field constraint (2.25) and its Bianchi identities (2.28). What remains to be shown is that the same is true for the matter multiplet. Again we start from the multiplet (ϕ^I,$\psi_{\alpha\dot{A}}$), satisfying the e.o.m. (2.43) and construct superfields out of it according to the recursions (2.40). It is convenient to introduce again the constraint functions

$$\begin{aligned} \mathcal{C}^I_{\alpha A} &:= \nabla_{\alpha A}\Phi^I - i\Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}} \;, \\ \mathcal{C}_{\alpha\beta A\dot{A}} &:= \nabla_{\alpha A}\Psi_{\beta\dot{A}} - \Gamma^I_{A\dot{A}}\nabla_{\alpha\beta}\Phi^I \;, \end{aligned} \tag{2.46}$$

where we used the same symbols as for the free matter multiplet, which now encode the minimally coupled constraints (2.37), (2.38) (but this should not lead to any confusion). Acting with $\epsilon^{\alpha A}Q_{\alpha A}$ on the recursions (2.40) one obtains the recursions for the susy transformed fields as

$$\begin{aligned} \mathcal{R}\delta\Phi^I &= i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\delta\Psi_{\alpha\dot{A}} + \Lambda\cdot\Phi^I - \epsilon^{\alpha A}\mathcal{C}^I_{\alpha A} \;, \\ \mathcal{R}\delta\Psi_{\beta\dot{A}} &= \theta^{\alpha A}\Gamma^I_{A\dot{A}}\nabla_{\alpha\beta}\delta\Phi^I + \Lambda\cdot\Psi_{\beta\dot{A}} - \epsilon^{\alpha A}\mathcal{C}_{\alpha\beta A\dot{A}} \;. \end{aligned} \tag{2.47}$$

Therefore, modulo super gauge transformations with the parameter Λ of (2.34) the recursion relations are susy covariant in case that the matter constraints (2.37), (2.38) and the Bianchi identities (2.28) are satisfied.

The rest of the proof that the component e.o.m. (2.43) imply superfield e.o.m. and superfield constraints, proceeds exactly as in the previous discussion of section 2.1 by simply replacing all derivative operators by covariant derivatives. Thus again, the superfield constraints are completely equivalent to the component multiplet with the e.o.m. (2.43). We will see in the next section how deformations of the constraint (2.25) will modify these results and introduce non-trivial interactions.

3. Interacting theories

3.1 Vector superfield with a modified constraint

In this section, we consider the vector superfields $\mathcal{A}_{\alpha A}$, $\mathcal{A}_{\alpha\beta}$ for which the constraint (2.25) is modified to

$$\{\nabla_{\alpha A}, \nabla_{\beta B}\} = 2i\left(\delta_{AB}\nabla_{\alpha\beta} + \varepsilon_{\alpha\beta}W_{AB}\right) \ , \tag{3.1}$$

where $W_{AB} = -W_{BA}$ is an antisymmetric $SO(8)$-tensor. There are two different situations in which the system (3.1) may appear. First, if W_{AB} is a given function of the matter superfields of the theory, i.e. $W_{AB} = W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$, the system (3.1) describes a *deformation* of the original constraint (2.25) which will in particular induce a (non-linear) deformation of the original (super)field equations of motion (2.43) by terms containing W_{AB} and its (super-)derivatives. This is the scenario we will be dealing with in this paper. As we will see, as soon as the matter superfields are coupled to the gauge superfields, W_{AB} is necessarily a function of them. In this case we will refer to the $SO(8)$-tensor W_{AB} as the *deformation potential*.

Alternatively, one might consider the vector multiplet independently and regard W_{AB} as an independent field defined by equation (3.1), in which case this equation rather amounts to parametrizing a *weakening* of the original constraint (2.25) to

$$\{\nabla_{\alpha A}, \nabla_{\beta B}\}\Big|_{(\mathbf{3},\mathbf{35_s})} = 0 \ . \tag{3.2}$$

In that case, the dynamics induced by (3.2) can be considered independently of the matter sector and will in particular lead to a different number of degrees of freedom contained in the vector superfield.

In either case the Bianchi identities impose conditions on W_{AB} for the the constraint (3.1) being self consistent.

Bianchi identities

As in the free theory, the immediate nontrivial conditions on the superfields are given by the first two Bianchi identities in (2.26), where (3.1) appears. Using the constraint (3.1) the first Bianchi identity imposes the condition

$$\begin{aligned}\delta_{AB}\mathcal{F}_{\alpha\beta,\gamma C} + \delta_{CA}\mathcal{F}_{\gamma\alpha,\beta B} + \delta_{BC}\mathcal{F}_{\beta\gamma,\alpha A} = \\ \varepsilon_{\beta\gamma}\nabla_{\alpha A}W_{BC} + \varepsilon_{\gamma\alpha}\nabla_{\beta B}W_{CA} + \varepsilon_{\alpha\beta}\nabla_{\gamma C}W_{AB} \ .\end{aligned} \tag{3.3}$$

Decomposing the terms of this equation analogously to (2.8) according to their $SO(8)$ representation content, one deduces that solvability requires the $\mathbf{160_s}$ to vanish within the the tensor product $\nabla_{\alpha A} W_{BC} \sim \mathbf{8_s} \otimes \mathbf{28} = \mathbf{8_s} \oplus \mathbf{56_s} \oplus \mathbf{160_s}$. This implies the existence of superfields $\lambda_{\alpha A}$, in the $\mathbf{8_s}$, and $\rho_{\alpha ABC} = \rho_{\alpha [ABC]}$, in the $\mathbf{56_s}$, such that the superderivative $\nabla_{\alpha A} W_{BC}$ satisfies the condition[9]

$$\nabla_{\alpha A} W_{BC} \Big|_{\mathbf{160_s}} = 0 \qquad \Longrightarrow \qquad \nabla_{\alpha A} W_{BC} = \delta_{A[B} \lambda_{C]\alpha} + \rho_{\alpha ABC} \ . \tag{3.4}$$

This constraint will play a central role in the following. In particular, if we consider W_{AB} as a function of the matter fields of the theory, this composite superfield must satisfy (3.4) in order for the system (3.1) to be consistent. The Bianchi identity (3.3) then fixes the fermionic field strength $\mathcal{F}_{\alpha\beta,\gamma A}$ to

$$\mathcal{F}_{\alpha\beta,\gamma A} = -\varepsilon_{\gamma(\alpha} \lambda_{\beta)A} \ . \tag{3.5}$$

Using the constraint (3.1), the second Bianchi identity in (2.26) writes as

$$\nabla_{\alpha A} \mathcal{F}_{\gamma\delta,\beta B} + \nabla_{\beta B} \mathcal{F}_{\gamma\delta,\alpha A} = 2i(\delta_{AB} \mathcal{F}_{\gamma\delta,\alpha\beta} + \varepsilon_{\alpha\beta} \nabla_{\gamma\delta} W_{AB}) \ , \tag{3.6}$$

and with (3.5) implies the existence of another superfield $V_{AB} = V_{[AB]}$ in the $\mathbf{28}$, such that

$$\nabla_{\alpha A} \lambda_{\beta B} = i(\delta_{AB} \mathcal{F}_{\alpha\beta} + 2\nabla_{\alpha\beta} W_{AB} + \varepsilon_{\alpha\beta} V_{AB}) \ . \tag{3.7}$$

Here, $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{(\alpha\beta)}$ denotes the vector dual to the bosonic field strength, i.e. $\mathcal{F}_{\alpha\beta} := \varepsilon^{\gamma\delta} \mathcal{F}_{\alpha\gamma,\beta\delta}$. This duality is characteristic for three dimensions and we will use this relation frequently in the following.

The first Bianchi identity identifies $\mathcal{F}_{\alpha\beta,\gamma A}$ with a single field (3.5) and thus, contrary to the free case (2.28), also the third Bianchi identity in (2.26) gives a nontrivial condition on the superfields:

$$\nabla_{\alpha A} \mathcal{F}_{\beta\gamma} = \nabla_{\alpha(\beta} \lambda_{\gamma)A} + \varepsilon_{\alpha(\beta} \nabla_{\gamma)\delta} \lambda^{\delta}_{A} \ . \tag{3.8}$$

The equations (3.4), (3.5), (3.7) and (3.8) are the consistency conditions for the constraint (3.1), which are imposed by the Bianchi identities.

Deformed super-CS e.o.m. In the case that a deformation potential $W_{AB} = W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$ is chosen the derived superfields $\lambda_{\alpha A}$, $\rho_{\alpha ABC}$, etc. are also given functions of the matter superfields. In particular defines (3.7) the super field strength $\mathcal{F}_{\alpha\beta}$ in terms of the matter superfields in the following form:

$$\mathcal{E}_{\alpha\beta} := \mathcal{F}_{\alpha\beta} + \tfrac{i}{8}\, \nabla^{A}_{(\alpha} \lambda_{\beta)A} = \mathcal{F}_{\alpha\beta} - \tfrac{i}{28}\, \nabla^{A}_{(\alpha} \nabla^{B}_{\beta)} W_{AB} = 0 \ , \tag{3.9}$$

where we used (3.4) to express $\lambda_{\alpha A}$ in terms of the deformation potential W_{AB}. As in the free case (2.28) one obtains the superfield e.o.m. in the gauge sector from

[9] Symmetrization and antisymmetrization of indices is indicated by brackets ( ) and [ ], respectively, and is defined with total weight one, i.e. $x_{(\alpha\beta)} = \frac{1}{2}(x_{\alpha\beta} + x_{\beta\alpha})$, etc..

the second Bianchi identity and (3.9) explicitly shows, how the dynamics of the free Chern-Simons gauge field is deformed by the presence of the deformation potential W_{AB}.

A priori, with (3.9) the fourth Bianchi identity in (2.26), which takes the form

$$\nabla^{\alpha\beta}\mathcal{F}_{\alpha\beta} = 0 \ , \tag{3.10}$$

may give rise to yet another condition. However, one can evaluate the l.h.s. of (3.10) using the constraint (3.1) and the conditions (3.8), (3.5) to show that (3.10) is identically fulfilled and does not impose additional conditions.

Integrability conditions

The integrability conditions of the constraints derived from the Bianchi identities, in particular (3.4) and (3.7), determine the superderivatives of the various additional superfields and eventually allow to define a closed recursive system for a systematic superfield expansion analogous to the procedure in section 2. In the case that the gauge sector with the constraint (3.1) is considered as an independent system these are genuine conditions on these superfields which correspond to independent degrees of freedom. We give a thorough account on this scenario in appendix A.

By contrast, in choosing a certain deformation potential $W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$ satisfying the conditions (3.4),(3.7) and (3.8), the "sources" on the r.h.s are derived from W_{AB} and the integrability conditions are identically satisfied and give identities rather than conditions. In addition, the constraints (3.1) and (3.5) define $\mathcal{R}\mathcal{A}_{\alpha A}$ and $\mathcal{R}\mathcal{A}_{\alpha\beta}$ in terms of the matter superfields and thus form together with $\mathcal{R}\Phi^I$, $\mathcal{R}\Psi_{\alpha\dot{A}}$ a closed recursive system. We will carry out the detailed analysis of the superfield expansion, component equations and the equivalence thereof to the constraints in the next subsection, where we study the coupling between the gauge and matter sector.

We develop here the system of integrability conditions till the point we will need it for a general discussion of the possible couplings to the matter sector. Especially we want to clarify here which of the restrictions (3.4),(3.7), (3.8) on the choice for the deformation potential $W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$ are independent.

The integrability condition of (3.7) gives $\nabla_{\alpha A}V_{BC}$ and reproduces the third Bianchi identity (3.8). Analyzing the integrability conditions of (3.4) determines $\nabla_{\alpha A}\rho_{\beta BCD}$ and reproduces the second Bianchi identity (3.7) with $\mathcal{F}_{\alpha\beta}$ as given by the CS-e.o.m. (3.9). Consequently, the only remaining restriction on the choice of $W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$ is the condition (3.4).

The resulting covariant super derivatives of the various fields are:

$$\begin{aligned}
\nabla_{\alpha A}\,\rho_{\beta BCD} &= 3i\nabla_{\alpha\beta}W_{[BC}\delta_{D]A} - \tfrac{3i}{2}\varepsilon_{\alpha\beta}\delta_{A[B}V_{CD]} + 3i\varepsilon_{\alpha\beta}\left[W_{A[B}, W_{CD]}\right] + iU_{\alpha\beta\,ABCD} \ , \\
\nabla_{\alpha A}V_{BC} &= 2\varepsilon^{\beta\gamma}\nabla_{\alpha\beta}\left(\delta_{A[B}\lambda_{C]\gamma} - \rho_{\gamma ABC}\right) - \left[W_{BC}, \lambda_{A\alpha}\right] - 4\left[W_{A[B}, \lambda_{C]\alpha}\right] \ , \\
\nabla_{\alpha A}U_{\beta\gamma\,BCDE} &= 8\delta^{A[B}\nabla_{\alpha(\beta}\rho_{\gamma)}^{\ CDE]} - 4\delta^{A[B}\nabla_{\beta\gamma}\rho_{\alpha}^{\ CDE]} + \tau_{\alpha\beta\gamma\,ABCDE} \\
&\quad +4\varepsilon_{\alpha(\beta}\left(\tfrac{4}{3}[W^{A[B}, \rho_{\gamma)}^{\ CDE]}] - [W^{[BC}, \rho_{\gamma)}^{\ DE]A}] + 3\delta^{A[B}[W^{CD}, \lambda_{\gamma)}^{E]}]\right) \ ,
\end{aligned} \tag{3.11}$$

where the last equation for the superfield $U_{\beta\gamma BCDE} = U_{(\beta\gamma)[BCDE]}$ has been obtained from the integrability condition for the $\nabla_{\alpha A}\,\rho_{\beta BCD}$ equation. At this point superderivatives of the fields are determined up to the tensor $\tau_{\alpha\beta\gamma\,ABCDE} = \tau_{(\alpha\beta\gamma)\,[ABCDE]}$. This is all we need for a general discussion of the matter couplings and we refer to appendix A to see how the system closes.

We have thus shown, that deforming the free constraint (2.25) by choosing W_{AB} to be a certain function $W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$ of the matter superfields, the Bianchi identities are satisfied provided that W_{AB} satisfies the constraint (3.4). The super field strengths are given by (3.5) and the deformed super Chern-Simons equations (3.9). Consequently, the constraint (3.4) is the only condition on the choice of W_{AB} for the deformation (3.1) to be self-consistent.

3.2 Matter superfields and gauge matter coupling

In this section we study the consequences of the deformation (3.1) for the matter sector and give a detailed discussion parallel to the sections 2.2 and 2.3 of the coupled system regarding component field equations, supersymmetry transformations and the equivalence thereof to the combined constraint system. As for the gauge sector the deformation will modify the dynamics by terms polynomial in the deformation potential W_{AB} and its (super-)derivatives. Compatibility of the system will require W_{AB} to satisfy additional algebraic constraints.

Superfield constraints

The most conceivable starting point for the matter sector is to keep the covariantized constraint (2.37) for the scalar superfield Φ^I

$$\nabla_{\alpha A}\Phi^I = i\Gamma^I_{A\dot{A}}\,\Psi_{\alpha\dot{A}}\ , \tag{3.12}$$

and deduce the consequences due to the new vector superfield constraint (3.1). For a given constraint in the gauge sector, (3.12) to a large extent determines the resulting dynamics of the system.

Using the gauge field constraint (3.1) the integrability condition of (3.12) is now modified to

$$2\ \delta_{AB}\nabla_{\alpha\beta}\Phi^I + 2\ \varepsilon_{\alpha\beta}\,W_{AB}\cdot\Phi^I = \Gamma^I_{B\dot{A}}\nabla_{\alpha A}\Psi^{\dot{A}}_{\beta} + \Gamma^I_{A\dot{A}}\nabla_{\beta B}\Psi^{\dot{A}}_{\alpha}\ . \tag{3.13}$$

Repeating the analysis of section 2 determines $\nabla_{\alpha A}\Psi_{\beta\dot{A}}$ but also gives restrictions on the new (second) term on the l.h.s. Since the $\mathbf{160_v}$ in

$$W_{AB}\cdot\Phi^I \sim \mathbf{28}\otimes\mathbf{8_v} = \mathbf{8_v}\oplus\mathbf{56_v}\oplus\mathbf{160_v}\ , \tag{3.14}$$

is unpaired in equation (3.13) it has to vanish separately. In the following it will be often convenient to write W_{AB} in the vector notation $W_{IJ} = \frac{1}{4}\Gamma^{IJ}_{AB}W_{AB}$ (see appendix B), such that the constraint on $W_{AB} \cdot \Phi_K$ writes as

$$
\begin{aligned}
& W_{IJ} \cdot \Phi_K \Big|_{\mathbf{160_v}} = 0 \qquad \Longrightarrow \\
& \mathbb{P}_{160}^{[IJK]}(W_{IJ} \cdot \Phi_K) := W_{IJ} \cdot \Phi_K - W_{K[I} \cdot \Phi_{J]} + \tfrac{3}{7}\, \delta_{K[I} W_{J]L} \cdot \Phi^L \; = 0 \;. \qquad (3.15)
\end{aligned}
$$

In addition to the constraint (3.4) this will be the main restriction on the possible choices for the deformation potential $W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$, which fixes the details of the dynamics. In the following we will refer to these two constraints (3.4), (3.15), which determine the set of possible models, as the W*-constraints*. The algebraic W-constraint (3.15) also shows that as soon as the matter sector is coupled to the gauge sector, the modification W_{AB} of the gauge field constraint (3.1) has to be considered as a function of the the matter superfields which at least depends on Φ^I.

After some $SO(8)$-Γ-matrix algebra the integrability condition (3.13) yields

$$
\nabla_{\alpha A} \Psi_{\beta\dot{A}} = \Gamma^I_{A\dot{A}} \nabla_{\alpha\beta} \Phi^I + \tfrac{1}{2} \varepsilon_{\alpha\beta} \left(\tfrac{1}{7}\, \Gamma^I_{A\dot{A}} \delta^{JK} + \tfrac{1}{6}\, \Gamma^{IJK}_{A\dot{A}} \right) W_{IJ} \cdot \Phi_K \;, \qquad (3.16)
$$

for the superderivative of the fermionic superfield. Using the gauge field constraint (3.1) to express x-space covariant derivatives through covariant superderivatives and the various constraint relations and Bianchi identities of this section, one obtains the superfield equations for $\Psi_{\alpha\dot{A}}$ and Φ^I:

$$
\begin{aligned}
\mathcal{E}_{\alpha\dot{A}} :=\; & \varepsilon^{\beta\gamma} \nabla_{\alpha\beta} \Psi_{\gamma\dot{A}} \\
& + \tfrac{3}{14}\, W_{\dot{A}\dot{B}} \cdot \Psi_{\alpha\dot{B}} + \tfrac{3i}{16}\, \Gamma^I_{A\dot{A}}\, \lambda_{\alpha A} \cdot \Phi^I + \tfrac{i}{336}\, \Gamma^{ABC}_{I\dot{A}} \rho_{\alpha ABC} \cdot \Phi^I = 0 \;\;, \\
\mathcal{E}^I :=\; & \nabla^2 \Phi^I - \tfrac{1}{8} \left(3\, \Gamma^I_{A\dot{A}}\; \lambda_{\alpha A} \cdot \Psi^\alpha_{\dot{A}} + \tfrac{1}{21}\, \Gamma^{ABC}_{I\dot{A}} \rho_{\alpha ABC} \cdot \Psi^\alpha_{\dot{B}} \right) \\
& + \tfrac{3}{14}\, V^{IJ} \cdot \Phi^J - \tfrac{2}{49}\, W_{IJ} \cdot (W_{JK} \cdot \Phi^K) - \tfrac{1}{28}\, W_{JK} \cdot (W_{JK} \cdot \Phi^I) = 0 \;,
\end{aligned}
\qquad (3.17)
$$

where $V^{IJ} := \frac{1}{4}\, \Gamma^{IJ}_{AB}\, V_{AB}$, $W_{\dot{A}\dot{B}} := \frac{1}{4}\, \Gamma^{IJ}_{\dot{A}\dot{B}} W_{IJ}$, are special cases of $SO(8)$ triality relations. In the same spirit we have defined the symbol $\Gamma^{ABC}_{I\dot{A}} := \Gamma^{IJ}_{[AB} \Gamma^J_{C]\dot{A}}$, see appendix B for more details and several Γ-matrix identities which were employed in this calculation. Using the algebraic W-constraint (3.15) one can recast the scalar self-interaction involving W_{IJ} in different forms. Equations (3.17) together with (3.9) constitute the complete set of superfield e.o.m.

Superfield expansion

We again impose the "transverse" gauge (2.29) to construct the superfield expansion via a recursive system. Contracting the constraints for the matter fields (3.12), (3.16) and the gauge field constraint (3.1) with $\theta^{\alpha A}$, and the Bianchi identity (3.5) with

$\theta^{\gamma A}$, one obtains the recursion relations for the superfields Φ^I, $\Psi_{\alpha\dot{A}}$, $\mathcal{A}_{\alpha A}$ and $\mathcal{A}_{\alpha\beta}$:

$$\begin{aligned}
\mathcal{R}\Phi^I &= i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}} \ , \\
\mathcal{R}\Psi_{\beta\dot{A}} &= \theta^{\alpha A}\Gamma^I_{A\dot{A}}\nabla_{\alpha\beta}\Phi^I + \tfrac{1}{2}\ \theta^{\alpha A}\varepsilon_{\alpha\beta}\left(\tfrac{1}{7}\,\Gamma^I_{A\dot{A}}\delta^{JK} + \tfrac{1}{6}\,\Gamma^{IJK}_{A\dot{A}}\right)W_{IJ}\cdot\Phi_K \ , \\
(1+\mathcal{R})\,\mathcal{A}_{\beta B} &= 2i(\theta^\alpha_B\ \mathcal{A}_{\alpha\beta} + \theta^{\alpha A}\varepsilon_{\alpha\beta}W_{AB}) \ , \\
\mathcal{R}\mathcal{A}_{\alpha\beta} &= \theta^{\gamma A}\varepsilon_{\gamma(\alpha}\lambda_{\beta)A} \ ,
\end{aligned} \tag{3.18}$$

which generalize the recursions of the free theory (2.30) and (2.40). The composite superfields of the gauge sector, such as $\lambda_{\alpha A}$, $\rho_{\alpha ABC}$, V_{AB}, etc., are now given functions of the matter superfields via the deformation potential $W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$,

$$\begin{aligned}
&\lambda_{\alpha A} = \tfrac{2}{7}\ \nabla_{\alpha B}W_{BA} \ , \quad \rho_{\alpha ABC} = \nabla_{\alpha[A}W_{BC]} \ , \\
&V_{AB} = -\tfrac{i}{2}\ \varepsilon^{\alpha\beta}\nabla_{\alpha A}\nabla_{\beta C}W_{CB} \ , \quad \text{etc.},
\end{aligned} \tag{3.19}$$

as can be seen from equations (3.4), (3.7) and (3.11). The recursion relations for these composite superfields as well as for W_{AB} and $\mathcal{F}_{\alpha\beta}$ are determined by the recursions of the fundamental superfields (3.18), but on the constraint surface they are equivalently given by the contraction of (3.4), (3.7), (3.8) and (3.11) with $\theta^{\alpha A}$. Off the constraint surface, and thus when deriving the constraints from the component field equations, this is no longer true as we will see.

To second order in θ, the superfield expansion can be expressed in terms of the composite fields explicitly given in (3.19):

$$\begin{aligned}
\Phi^I &= \phi^I + i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\psi_{\alpha\dot{A}} + \tfrac{i}{2}\ \theta^{\alpha A}\theta^{\beta B}\ \Gamma^{IJ}_{AB}\ \overset{\circ}{\nabla}_{\alpha\beta}\phi^J \\
&\quad -\tfrac{i}{4}\ \theta^{\alpha A}\theta^{\beta B}\varepsilon_{\alpha\beta}\Big(\tfrac{1}{7}\ \delta_{AB}\ \overset{\circ}{W}_{IJ}\cdot\phi^J - \tfrac{1}{6}\ \Gamma^{ILMN}_{AB}\ \overset{\circ}{W}_{LM}\cdot\phi_N\Big) + \dots \ , \\
\Psi_{\beta\dot{A}} &= \psi_{\beta\dot{A}} + \theta^{\alpha A}\Gamma^I_{A\dot{A}}\overset{\circ}{\nabla}_{\alpha\beta}\phi^I \\
&\quad +\tfrac{1}{2}\ \theta^{\alpha A}\varepsilon_{\alpha\beta}(\tfrac{1}{7}\ \Gamma^I_{A\dot{A}}\delta^{JK} + \tfrac{1}{6}\ \Gamma^{IJK}_{A\dot{A}})\ \overset{\circ}{W}_{IJ}\cdot\phi_K + \dots \ , \\
\mathcal{A}_{\alpha\beta} &= A_{\alpha\beta} + \theta^{\gamma C}\varepsilon_{\gamma(\alpha}\overset{\circ}{\lambda}_{\beta)C} \\
&\quad +\tfrac{i}{2}\ \theta^{\gamma C}\theta^{\delta D}\left(\tfrac{1}{2}\varepsilon_{\gamma\delta}\ \delta_{CD}F_{\alpha\beta} - 2\varepsilon_{\gamma(\alpha}\overset{\circ}{\nabla}_{\beta)\delta}\overset{\circ}{W}_{CD} + \varepsilon_{\gamma(\alpha}\varepsilon_{\beta)\delta}\overset{\circ}{V}_{CD}\right) + \dots
\end{aligned} \tag{3.20}$$

while for $\mathcal{A}_{\beta B}$, the expansion of is formally the same as in abelian case (A.9) of appendix A, see also (A.11) for more details. To obtain explicit expressions one has to compute the lowest order components of the composite fields in (3.19). To this end we assume here and in the following that W_{AB} depends on Φ^I only and not on the fermionic superfield $\Psi_{\beta\dot{A}}$, i.e.

$$W_{AB} = W_{AB}\ (\Phi^I)\ . \tag{3.21}$$

The explicit cases that we are going to study in this work fall into this class of deformation potentials W_{AB}. Using (3.12) and with $\partial_{Ia} := \partial/\partial\phi^{Ia}$, where the index

a refers to the representation of the gauge (structure) group, the projection on the lowest components for the composite fields takes the form

$$
\begin{aligned}
&\overset{\circ}{W}_{AB} = W_{AB}(\phi) \ , \quad \overset{\circ}{\lambda}_{\alpha B} = \tfrac{2i}{7}\, \psi^a_{\alpha \dot{A}} \Gamma^I_{A\dot{A}}\ \partial_{Ia} \overset{\circ}{W}_{AB} \ , \quad \overset{\circ}{\rho}_{\alpha ABC} = i\psi^a_{\alpha\dot{A}}\ \partial_{Ia} \overset{\circ}{W}_{[BC} \Gamma^I_{A]\dot{A}} \ , \\
&\overset{\circ}{V}_{AB} = [\overset{\circ}{W}_{AC}, \overset{\circ}{W}_{CB}] - \varepsilon^{\alpha\beta} \psi^a_{\alpha\dot{A}} \psi^b_{\beta\dot{B}}\ \Gamma^I_{A\dot{A}} \Gamma^J_{C\dot{B}}\ \partial_{Ia}\partial_{Jb} \overset{\circ}{W}_{CB} \\
&\qquad\qquad + \tfrac{i}{7}\ (\overset{\circ}{W}_{IJ} \cdot \phi^J)^a\ \partial_{Ia} \overset{\circ}{W}_{AB} - \tfrac{i}{6} \Gamma^{ILMN}_{AC}\ (\overset{\circ}{W}_{LM} \cdot \phi_N)^a\ \partial_{Ia} \overset{\circ}{W}_{BC} \ . \qquad (3.22)
\end{aligned}
$$

With the above relations and the superfield expansions (3.20) one obtains from (3.17) in a straightforward way the component field e.o.m. for the component fields ϕ^I and $\psi_{\alpha\dot{A}}$. The CS-e.o.m. is the lowest component of (3.9) and can be computed analogously to (3.22). Together, the full system of component e.o.m. is given by

$$
\begin{aligned}
&\overset{\circ}{\mathcal{E}}_{\alpha\beta} = F_{\alpha\beta} + \frac{1}{28} \left(\Gamma^{IJ}_{AB} \overset{\circ}{\nabla}_{\alpha\beta} \phi^{Ia} \partial_{Ja} \overset{\circ}{W}_{AB} - i\psi^a_{\dot{A}(\alpha} \psi^b_{\beta)\dot{B}}\ \Gamma^I_{A\dot{A}} \Gamma^J_{B\dot{B}}\ \partial_{Ia}\partial_{Jb} \overset{\circ}{W}_{AB} \right) = 0 \ , \\
&\overset{\circ}{\mathcal{E}}_{\alpha\dot{A}} = 0 \ , \qquad \overset{\circ}{\mathcal{E}}^I = 0 \ . \qquad (3.23)
\end{aligned}
$$

The supersymmetry transformations for the independent component fields ϕ^I, $\psi_{\alpha\dot{A}}$ and $A_{\alpha\beta}$ are again obtained from the superfield expansion, (3.20), by acting with $\epsilon^{\alpha A} Q_{\alpha A}$ and modding out a restoring super gauge transformation with gauge parameter

$$
\Lambda = \epsilon^{\alpha A} \mathcal{A}_{\alpha A} = i\epsilon^{\alpha A} \theta^{\beta B} (\delta_{AB} A_{\alpha\beta} + \varepsilon_{\alpha\beta} \overset{\circ}{W}_{AB}) + \dots \ , \qquad (3.24)
$$

which is formally the same as for the free CS-multiplet (2.34) but has a more non-trivial superfield expansion, see (A.9). The obtained component supersymmetry transformations are,

$$
\begin{aligned}
\delta\phi^I &= i\epsilon^{\alpha A}\ \Gamma^I_{A\dot{A}} \psi_{\alpha\dot{A}} \ , \\
\delta\psi_{\beta\dot{A}} &= \epsilon^{\alpha A} \Big(\Gamma^I_{A\dot{A}} \overset{\circ}{\nabla}_{\alpha\beta} \phi^I + \tfrac{1}{2}\ \varepsilon_{\alpha\beta} \left(\tfrac{1}{7}\ \Gamma^I_{A\dot{A}} \delta^{JK} + \tfrac{1}{6}\ \Gamma^{IJK}_{A\dot{A}} \right) \overset{\circ}{W}_{IJ} \cdot \phi_K \Big) \ , \\
\delta A_{\alpha\beta} &= \tfrac{2i}{7}\ \epsilon^{\gamma B} \varepsilon_{\gamma(\alpha} \psi^a_{\beta)\dot{A}}\ \Gamma^I_{A\dot{A}}\ \partial_{Ia} \overset{\circ}{W}_{AB} \ , \qquad (3.25)
\end{aligned}
$$

and again resemble the recursion relations of the associated superfields (3.18). Equations (3.23) and (3.25) show how the deformation potential W_{AB} modifies the dynamics and supersymmetry transformations of the component fields compared to the minimally coupled free CS-multiplet (2.43) and (2.45). The deformation potential $\overset{\circ}{W}_{AB} = W_{AB}(\phi^I)$ cannot be chosen arbitrarily but inherits the lowest components of the W-constraints (3.4) and (3.15). These conditions are also necessary for the component field equations (3.23) to be invariant under the supersymmetry transformations (3.25). The algebraic W-constraints (3.15) is the same for the lowest component fields, since it just constrains the functional form of $W_{AB}(\phi^I)$. The lowest component of the differential W-constraint (3.4) is straightforwardly obtained by using (3.12). Together, one finds for $\overset{\circ}{W}_{AB} = W_{AB}(\phi^I)$ the conditions

$$\mathbb{P}_{160}^{[IJK]}(\overset{\circ}{W}_{IJ} \cdot \phi_K) = 0 \ , \qquad \mathbb{P}_{160}^{[ABC]}(\Gamma^I_{A\dot{A}}\partial_{Ia}\overset{\circ}{W}_{BC}) = 0 \ \ , \tag{3.26}$$

where the projector $\mathbb{P}_{160}^{[RST]}$, acting on three indices R, S, T referring to the same representation, was introduced in (3.15).

Equivalence to component e.o.m.

In this part we prove that the component multiplet $(\phi^I, \psi_{\alpha\dot{A}}, A_{\alpha\beta})$ satisfying the e.o.m. (3.23) with the conditions (3.26) for the deformation potential, and the supersymmetry transformations (3.25) is equivalent to our constraint system, in particular the gauge field constraint (3.1) and the matter field constraint (3.12) and consequently their Bianchi identities and integrability conditions. The reader who is only interested in the mere fact of this equivalence may skip the details of the proof presented here.

As in the previous sections we construct superfields $\mathcal{A}_{\alpha A}$, $\mathcal{A}_{\alpha\beta}$, Φ^I and $\Psi_{\alpha\dot{A}}$ out of the component multiplet according to the recursion relations[10] (3.18). One can ask again if this definition of superfields is susy covariant and mutatis mutandis one obtains the result analogous to (2.33) and (2.47) that these superfields are susy covariant modulo supergauge transformations with the parameter (3.24) if the constraints (3.1), (3.5), (3.12) and (3.16) are satisfied.

To demonstrate the equivalence between component field equations and the constraints we again construct a recursive system for the constraints- and superfield e.o.m. expressions. Due to the non-trivial coupling of the gauge and matter sector, and in particular due to the conditions on the deformation potential W_{AB}, the situation is quite involved and we introduce a more symbolic notation such that the structure of the system remains clear. From the gauge sector the following expressions, resembling (3.1), (3.5), (3.7) and (3.4) will occur in the recursive system

$$\begin{aligned} G^{(1)} = G_{\alpha A,\beta B} &:= \{\nabla_{\alpha A}, \nabla_{\beta B}\} - 2i(\delta_{AB}\nabla_{\alpha\beta} + \varepsilon_{\alpha\beta}W_{AB}) \ , \\ G^{(2)} = G_{\alpha\beta,\gamma A} &:= \mathcal{F}_{\alpha\beta,\gamma A} + \varepsilon_{\gamma(\alpha}\lambda_{\beta)A} \ , \\ \mathcal{E}^{\text{cs}} = \mathcal{E}_{\alpha\beta} &:= \mathcal{F}_{\alpha\beta} - X_{\alpha\beta} \ , \\ G = G_{\alpha A,BC} &:= \nabla_{\alpha A}W_{BC} - (\delta_{A[B}\lambda_{C]\alpha} + \rho_{\alpha ABC}) \ , \end{aligned} \tag{3.27}$$

where we have introduced the abbreviation $X_{\alpha\beta} = -\frac{i}{8}\nabla^C_{(\alpha}\lambda_{\beta)C}$, and the other composite fields were given in (3.19). The expressions of the matter sector, resembling

[10]We do not intend to carry this out explicitly but use the recursions (3.18) as an implicit definition of the superfields. The explicit calculation would be rather messy, especially since off the constraint surface one cannot use the previously given recursion relations for the composite fields, as we will demonstrate now.

(3.12), (3.16), (3.17) and (3.15), are:

$$
\begin{aligned}
C^{(1)} = C^I_{\alpha A} &:= \nabla_{\alpha A}\Phi^I - i\Gamma^I_{A\dot A}\Psi_{\alpha\dot A}\ ,\\
C^{(2)} = C_{\alpha A,\beta\dot A} &:= \nabla_{\alpha A}\Psi_{\beta\dot A} - \Gamma^I_{A\dot A}\nabla_{\alpha\beta}\Phi^I\\
&\qquad\qquad -\tfrac{1}{2}\varepsilon_{\alpha\beta}\left(\tfrac{1}{7}\ \Gamma^I_{A\dot A}\delta^{JK} + \tfrac{1}{6}\ \Gamma^{IJK}_{A\dot A}\right) W_{IJ}\cdot\Phi_K\ ,\\
\mathcal{E}^{\rm ferm} &:= \mathcal{E}_{\alpha\dot A}\ ,\qquad \mathcal{E}^{\rm bos} := \mathcal{E}^I\ ,\\
C_{IJK} &:= \mathbb{P}^{[IJK]}_{160}(W_{IJ}\cdot\Phi_K)\ .
\end{aligned}
\tag{3.28}
$$

The explicit expressions for $\mathcal{E}_{\alpha\dot A}$, $\mathcal{E}^I$ were given in (3.17). In the following, the detailed index structure of the occurring expressions will not be important and in general we stick to the notation on the l.h.s of these definitions.

To determine the action of the recursion operator $\mathcal{R}$ (2.29) on the expressions (3.27), (3.28) we will need the superderivatives of the composite fields off the constraint surface, i.e. the analogs of (3.7), (3.8) and (3.11), but with $\mathcal{F}_{\alpha\beta}$ replaced with $X_{\alpha\beta}$[11]. These equations were obtained as consecutive integrability conditions of the differential W-constraint (3.4). Off the constraint surface one has to start instead from G in (3.27). Keeping track also of the other constraints one finds the following modifications of (3.7), (3.8), (3.11):

$$
\begin{aligned}
\nabla_{\alpha A}\lambda_{\beta B} &\to \nabla_{\alpha A}\lambda_{\beta B} + \{G^{(1)}W + \nabla G\}\ \ ,\\
\nabla_{\alpha A}\rho_{\beta BCD} &\to \nabla_{\alpha A}\rho_{\beta BCD} + \{G^{(1)}W + \nabla G\}\ \ ,\\
\nabla_{\alpha A}X_{\beta\gamma} &\to \nabla_{\alpha A}X_{\beta\gamma} + \{G^{(1)}\lambda + \nabla(G^{(1)}W+\nabla G) + G^{(2)}W\}\ \ ,\\
\nabla_{\alpha A}V_{BC} &\to \nabla_{\alpha A}V_{BC} + \{G^{(1)}\lambda + \nabla(G^{(1)}W+\nabla G) + G^{(2)}W\}\ \ ,\\
\nabla_{\alpha A}U_{\beta\gamma BCDE} &\to \nabla_{\alpha A}U_{\beta\gamma BCDE} + \{G^{(1)}\lambda + \nabla(G^{(1)}W+\nabla G) + G^{(2)}W + GW\}\ ,
\end{aligned}
\tag{3.29}
$$

where ∇ symbolically stands for a superderivative $\nabla_{\alpha A}$ with unspecified indices. In addition we need the expression for $\mathcal{R}\mathcal{F}_{\alpha\beta}$. As a consequence of the recursive definitions of the independent superfields (3.18) certain contractions of the constraints with $\theta^{\alpha A}$ vanish identically, i.e. $\theta^{\alpha A}C^I_{\alpha A} = \theta^{\alpha A}C_{\alpha A,\beta\dot A} = \theta^{\alpha A}G_{\alpha A,\beta B} = \theta^{\delta D}G_{\alpha\beta,\delta D} = 0$. With this one finds by acting with $\nabla_{\gamma\delta}$ on $\mathcal{R}\mathcal{A}_{\alpha\beta}$ in (3.18)

$$
\mathcal{R}\mathcal{F}_{\alpha\beta} = \theta^{\delta D}(\nabla_{\delta(\alpha}\lambda_{\beta)D} + \varepsilon_{\delta(\alpha}\,\nabla_{\beta)\gamma}\lambda^\gamma_D)\ \ , \tag{3.30}
$$

which is the same as on the constraint surface, i.e. the equation obtained by contraction of (3.8) with $\theta^{\alpha A}$.

For the W-constraints the results follow directly from the conditions (3.26) on the lowest components of the deformation potential W_{AB}. The algebraic W-constraint in (3.28) is identically zero, i.e. $C_{IJK} = 0$, due to the first condition in (3.26). The last

[11] Note that on the constraint surface means also $\mathcal{F}_{\alpha\beta} = X_{\alpha\beta}$ with $X_{\alpha\beta}$ given below (3.27), i.e. the CS-e.o.m., see also the discussion above (3.11).

equation in (3.27) can be read as $G = \mathbb{P}_{160}^{[ABC]}(\nabla_{\alpha A} W_{BC})$ and thus as a consequence of the second condition in (3.26) the differential W-constraint takes the form

$$G_{\alpha A,BC} = \mathbb{P}_{160}^{[ABC]}(C_{\alpha A}^{Ia}\partial_{Ia}W_{BC}) \quad \text{or} \quad G \sim C^{(1)}\partial W \ , \tag{3.31}$$

where the second expression is of the symbolic form that we will use in this section.

We now have all the ingredients needed to compute the action of $\mathcal{R}$ on the other expressions (3.27) and (3.28). Using the recursions (3.18) and the relations derived in this part, one finds for the gauge sector

$$\begin{aligned}
(2+\mathcal{R})\ G^{(1)} &\sim \theta\{G^{(2)} + G\} \ , \\
(1+\mathcal{R})\ G^{(2)} &\sim \theta\{\mathcal{E}^{\rm cs} + (G^{(1)}W + \nabla G)\} \ , \\
\mathcal{R}\ \mathcal{E}^{\rm cs} &\sim \theta\{G^{(1)}\lambda + G^{(2)}W + \nabla(G^{(1)}W + \nabla G)\} \ ,
\end{aligned} \tag{3.32}$$

which are obtained more or less straightforwardly. For the matter sector one obtains

$$\begin{aligned}
(1+\mathcal{R})\ C^{(1)} &\sim \theta C^{(2)} \ , \\
(1+\mathcal{R})\ C^{(2)} &\sim \theta\{\mathcal{E}^{\rm ferm} + \nabla_{\alpha\beta}C^{(1)} + G^{(2)}\Phi + (G\Phi + WC^{(1)})\} \ , \\
\mathcal{R}\ \mathcal{E}^{\rm ferm} &\sim \theta\{\mathcal{E}^{\rm cs}\Phi + \mathcal{E}^{\rm bos} + \nabla(G\Phi + WC^{(1)}) + (G^{(1)}W\Phi + WC^{(2)})\} \ , \\
\mathcal{R}\ \mathcal{E}^{\rm bos} &\sim \theta\{\nabla_{\alpha\beta}\mathcal{E}^{\rm ferm} + \mathcal{E}^{\rm cs}\Psi + \nabla\nabla(G\Phi + WC^{(1)}) \\
&\quad + \nabla(G^{(1)}W\Phi + WC^{(2)}) + (G^{(2)}W\Phi + GW\Phi + WWC^{(1)})\} \ ,
\end{aligned} \tag{3.33}$$

where $\nabla_{\alpha\beta}$ symbolically stands for a bosonic covariant derivative, the given indices have no specific meaning. The first relation in (3.33) is straightforwardly obtained and uses the fact that $C_{IJK} = 0$, as explained above (3.31). The derivation of the other relations is rather involved and uses, in this order, the first, second and third superderivative of the just mentioned relation, i.e. $\nabla C_{IJK} = 0$, $\nabla\nabla C_{IJK} = 0$ and $\nabla\nabla\nabla C_{IJK} = 0$. Via (3.29) these produce a number of constraints which we extracted here, the remaining terms are found to cancel with the help of an algebraic computation using Mathematica.

The notation used in (3.32) and (3.33) is rather formal, the suppressed index structure appears in all kind of combinations. This is enough information to show recursively that the whole system of constraints (3.27), (3.28) vanishes to all orders in θ as a consequence of the equations for the component fields (3.23), (3.26). In the first step one sees that to lowest order all expressions in (3.27) and (3.28) are zero due to (3.23), (3.26) or the recursion relations (3.32), (3.33):

$$\overset{\circ}{C}{}^{(1)} = \overset{\circ}{G} = \overset{\circ}{G}{}^{(1)} = \overset{\circ}{C}{}^{(2)} = \overset{\circ}{G}{}^{(2)} = \overset{\circ}{\mathcal{E}}{}^{\rm ferm} = \overset{\circ}{\mathcal{E}}{}^{\rm cs} = \overset{\circ}{\mathcal{E}}{}^{\rm bos} = 0 \ . \tag{3.34}$$

In the sequence given here for the lowest component it is easy to show using (3.32), (3.33) and (3.31), that to order $(n+1)$ in θ all expressions in (3.27), (3.28) are zero if they vanish at order n (the only subtlety one has to be careful about is the appearance of the superderivatives ∇ in (3.27), (3.28), which brings in higher order coefficients). With (3.34) this inductively proves that all expressions in (3.27), (3.28), vanish to all orders in θ due to the component field equations (3.23), (3.26), and thus shows the equivalence of the component field formulation and the constraints (3.1) and (3.12) and all their consequences.

Concluding this section, we have shown that the weaker gauge field constraint (3.1) is consistent only if the deformation potential W_{AB} satisfies the differential W-constraint (3.4). Coupling to the matter system via the same constraint as in the free CS case, (3.12), further imposes the algebraic W-constraint (3.15) on W_{AB} and thus necessarily makes the deformation potential a function of the matter superfields. This results in the interacting CS- and matter superfield e.o.m. (3.9) and (3.17). For the case that W_{AB} is a function exclusively of the scalar superfield Φ^I we gave the component field e.o.m. and the supersymmetry transformations (3.23), (3.25), (3.26) and demonstrated the equivalence of the component field equations to the superfield constraints. The generalization of these considerations to a more general deformation potential W_{AB}, depending also on the fermionic superfield $\Psi_{\alpha\dot{A}}$ is straightforward. In the next section we will give explicit solutions to the W-constraints which will imply the conformal BLG-model and $\mathcal{N} = 8$ SYM theory in its dual formulation, respectively.

4. Solutions to the superspace constraints

In this section, we present different solutions to the obtained superspace constraints and show how all known examples of three-dimensional $\mathcal{N} = 8$ gauge theories fit into our framework. Let us start by reviewing the structure of superspace constraints identified so far. The matter sector of these three-dimensional gauge theories is described by a scalar superfield subject to the constraint (3.12)

$$\nabla_{\alpha A}\, \Phi^I \Big|_{\mathbf{56_c}} = 0\,. \tag{4.1}$$

The full theory is then identified by specifying their gauge algebra $\mathfrak{g}$ (2.21) as a subalgebra of $\mathfrak{gl}(N, \mathbb{R}) \oplus_s \mathfrak{t}(8N)$ and by choosing $W_{AB}(\Phi^I, \Psi_{\alpha\dot{A}})$ in (3.1) as a function of the matter superfields of the theory. This choice of the deformation potential W_{AB} is not arbitrary but must satisfy two independent superfield conditions, the W-constraints (3.4) and (3.15):

$$\nabla_{\alpha A} W_{BC} \Big|_{\mathbf{160_s}} = 0\,, \tag{4.2}$$

$$W_{IJ} \cdot \Phi_K \Big|_{\mathbf{160_v}} = 0\,. \tag{4.3}$$

The first equation requires that the deformation potential W_{AB} depends on the matter fields such that (4.2) is satisfied as a consequence of (4.1). In contrast, equation (4.3) also explicitly contains the action of the gauge group on the matter fields and will thus put further restrictions on the possible gauge groups. We will see in explicit examples, that the conditions (4.2), (4.3) are truly independent as there are solutions to either one of them that do not solve the other equation.

4.1 Conformal gauge theories

In this section we consider gauge groups G that are subgroups of $GL(N, \mathbb{R})$, N being the number of scalar super-multiplets, such that the superfield $\mathcal{A}_{\alpha A}$ can be represented as a matrix in the adjoint representation of G. Accordingly, we label by indices $a, b, \ldots,$ the representation of G in which the matter superfields Φ_I and $\Psi_{\alpha\dot{A}}$ transform. Matter and gauge superfields are thus denoted as Φ_I^a, $\Psi^a_{\alpha\dot{A}}$, and $(\mathcal{A}_{\alpha A})^a{}_b$, $(\mathcal{A}_{\alpha\beta})^a{}_b$, respectively.

The constraint (3.1) implies that the composite field W_{AB} has canonical dimension one. Given that the scalar fields have canonical dimension one half in three dimensions, scale invariance implies that with a polynomial ansatz W_{AB} is bilinear in the scalar superfields Φ_I^a, with the most general ansatz given by

$$(W_{IJ})^a{}_b \equiv f^a{}_{b,cd}\, \Phi_I^c \Phi_J^d \,, \tag{4.4}$$

where the dimensionless constants $f^a{}_{b,cd}$ have to be antisymmetric in the last two indices, i.e. $f^a{}_{b,[cd]} = f^a{}_{b,cd}$. Gauge covariance requires that $f^a{}_{b,cd}$ is an invariant tensor of the gauge group G, and per construction W_{IJ} has to be an element of the Lie algebra and therefore $f^a{}_{b,dc} \in \mathfrak{g}$ for any d and c. Together, this translates into a quadratic condition for the tensor $f^a{}_{b,cd}$

$$f^g{}_{c,ab} f^e{}_{f,gd} - f^g{}_{d,ab} f^e{}_{f,gc} = f^g{}_{f,cd} f^e{}_{g,ab} - f^g{}_{f,ab} f^e{}_{g,cd} \,, \tag{4.5}$$

which can be obtained by explicitly evaluating the action of W_{IJ} on a W_{KL} and comparing this to the adjoint action by commutator. The same relation was obtained in [30] for the embedding tensor in a component field approach.

It is straightforward to check, that (4.4) is a solution to (4.2) as a consequence of (4.1): as $\nabla_{\alpha A} W_{BC}$ is composed of a single Φ^I and a single $\Psi_{\alpha\dot{A}}$, w.r.t. $SO(8)$ it transforms in the tensor product $\mathbf{8_v} \otimes \mathbf{8_c} = \mathbf{8_s} + \mathbf{56_s}$, which does not contain a $\mathbf{160_s}$. To solve the remaining constraint (4.3) we evaluate the action of (4.4) on a scalar field

$$(W_{IJ} \cdot \Phi_K)^a = f^a{}_{b,cd}\, \Phi_I^b \Phi_J^c \Phi_K^d \,. \tag{4.6}$$

This shows that the tensor $f^a{}_{b,cd}$ needs to satisfy complete antisymmetry in the last three indices $f^a{}_{b,cd} = f^a{}_{[bcd]}$, such that

$$(W_{IJ} \cdot \Phi_K)^a = f^a{}_{[bcd]}\, \Phi^b_{[I} \Phi^c_J \Phi^d_{K]} \,, \tag{4.7}$$

transforms in the $\mathbf{8_v}^{\otimes_{\rm alt}3} = \mathbf{56_v}$ of $SO(8)$, thus satisfying (4.3). For such a tensor $f^a{}_{b,cd}$, the quadratic equation (4.5) reduces to the so-called *fundamental identity.* The same condition on a tensor $f^a{}_{bcd}$, interpreted as a structure constants of a three-algebra, has been used in [31] in a component formulation of the equations of motion. This shows how the constructions of [1, 2, 31] are embedded into our superspace analysis. The existence of an action furthermore requires the existence of a metric h_{ab} and total antisymmetry of the tensor $f_{abcd} \equiv h_{ae} f^e{}_{[bcd]}$. It has been shown in a number of papers (see e.g. [32, 33]), that for a positive definite metric h_{ab}, equation (4.5) admits no other solutions than the compact $SO(4)$ of the original construction of [1, 2]. Solutions of (4.5) with indefinite metric have been found and studied in [34–36].

In order to complete the construction of this example, we evaluate the general formulae of the last section for the particular choice (4.4). From (3.19), we obtain

$$\begin{aligned}
(\lambda_{\alpha A})^a{}_b &= i f^a{}_{bcd}\, \Gamma^I_{A\dot{A}}\, \Psi^c_{\alpha\dot{A}} \Phi^d_I \,, \qquad (\rho_{\alpha\, ABC})^a{}_b \;=\; -\tfrac{1}{2} i f^a{}_{bcd}\, \Gamma^{ABC}_{I\dot{A}}\, \Psi^c_{\alpha\dot{A}} \Phi^d_I \,, \\
(V_{AB})^a{}_b &= -\tfrac{1}{2} i f^a{}_{bcd}\, \varepsilon^{\alpha\beta} \Gamma^{\dot{A}\dot{B}}_{AB}\, \Psi^c_{\alpha\dot{A}} \Psi^d_{\beta\dot{B}} + \tfrac{1}{4} f^a{}_{bcd} f^c{}_{efg}\, \Gamma^{IJ}_{AB}\, \Phi^e_I \Phi^f_J \Phi^g_K \Phi^d_K \,,
\end{aligned} \tag{4.8}$$

as well as the first order Chern-Simons equations of motion (3.9)

$$(\mathcal{F}_{\alpha\beta})^a{}_b = -f^a{}_{bcd} \Big(\Phi^c_I\, \nabla_{\alpha\beta} \Phi^d_I + i \Psi^c_{\alpha\dot{A}} \Psi^d_{\beta\dot{A}} \Big) \,. \tag{4.9}$$

This answers the question raised in [8], namely, how the in the Nambu-bracket realization by hand imposed "Chern-Simons-constraint" follows from consistency conditions of the scalar field equations.

After some calculation, the bosonic equations of motion (3.17) take the form

$$\nabla^2\, \Phi^a_I = \tfrac{i}{2}\, \varepsilon^{\alpha\beta} \Gamma^{IJ}_{\dot{A}\dot{B}}\, f^a{}_{bcd}\, \Psi^b_{\alpha\dot{A}} \Psi^c_{\beta\dot{B}} \Phi^d_J + \tfrac{1}{4}\, f^a{}_{bcd} f^b{}_{efg}\, \Phi^c_J \Phi^f_J \Phi^d_K \Phi^g_K \Phi^e_I \,, \tag{4.10}$$

and coincide with the result of [31]. For the theories with action, they exhibit the Yukawa couplings and the sextic scalar potential of [1].

4.2 Yang-Mills gauge theories

It has been shown in [37, 38] that three-dimensional Yang-Mills gauge theories have an equivalent formulation as matter-coupled Chern-Simons gauge theories with non-semisimple gauge group

$$G = G_{\rm YM} \ltimes \mathbb{T}_k \,, \tag{4.11}$$

where $\mathbb{T}_k$ denotes a set of $k \equiv \dim G_{\rm YM}$ translations, transforming in the adjoint representation of $G_{\rm YM}$. This allows to embed also Yang-Mills gauge theories into the general superspace formulation presented above. In the context of M2 branes, this duality has been discussed in [36, 39].

In order to realize (4.11) as a subgroup of $GL(N,\mathbb{R}) \ltimes \mathbb{T}(8N)$, we start from matter fields Φ^a_I, $\Psi^a_{\alpha\dot{A}}$ in the adjoint representation (thus $N = k$), with the index a now labelling the adjoint representation of the Yang-Mills gauge group $G_{\rm YM}$, and $f_{ab}{}^c$ denoting the Yang-Mills structure constants. To obtain the subalgebra $\mathfrak{t}$ associated with the subgroup $\mathbb{T}_k \subset \mathbb{T}(8k)$ we choose a fixed $SO(8)$-vector ξ_I and define the generators $\mathcal{T}_a$ of $\mathfrak{t}$ as

$$\mathcal{T}_a = \xi_I \mathcal{T}^I_a \ , \tag{4.12}$$

with a constant vector ξ_I, and where the $\mathcal{T}^I_a$ span the full Lie algebra of $\mathbb{T}(8k)$. The gauge superfields in the covariant derivatives as defined in (2.22) are thus chosen to be

$$\mathcal{A}_{\alpha A} = \mathcal{A}^M_{\alpha A} i T_M = \mathcal{A}^a_{\alpha A}\ iT_a + \mathcal{B}^a_{\alpha A}\ i\mathcal{T}_a =: \hat{\mathcal{A}}_{\alpha A} + \mathcal{B}_{\alpha A}\ , \tag{4.13}$$

with the Yang-Mills and the translation generators acting on the scalar superfield as

$$T_a \cdot \Phi^b_I = f_{ac}{}^b \Phi^c_I\ , \qquad \mathcal{T}_a \cdot \Phi^b_I = \xi_I \delta^b_a\ , \tag{4.14}$$

respectively. The constant vector ξ_I breaks $SO(8)$ down to $SO(7)$. The algebra of the generators (2.21) hence splits into the semidirect sum as

$$[T_M, T_N] = f_{MN}{}^K T_K \quad \leftrightarrow \quad \left\{ \begin{array}{l} [T_a, T_b] = f_{ab}{}^c T_c \\ [T_a, \mathcal{T}_b] = f_{ab}{}^c \mathcal{T}_c \\ [\mathcal{T}_a, \mathcal{T}_b] = 0 \end{array} \right. . \tag{4.15}$$

The bosonic gauge superfield $\mathcal{A}_{\alpha\beta} = \mathcal{A}^M_{\alpha\beta} T_M$ is decomposed analogously to (4.13), except for the factor of i (2.22). With regard to the separation of the gauge superfields we can write the covariant derivatives accordingly,

$$\nabla_{\alpha A} = \hat{\nabla}_{\alpha A} + \mathcal{B}_{\alpha A}\ , \quad \nabla_{\alpha\beta} = \hat{\nabla}_{\alpha\beta} + \mathcal{B}_{\alpha\beta}\ , \tag{4.16}$$

where $\hat{\nabla}_{\alpha A}$ contains only $\hat{\mathcal{A}}_{\alpha A}$, etc.. The action on the superfield Φ_I then takes the form

$$\nabla_{\alpha A} \Phi^a_I = \hat{\nabla}_{\alpha A} \Phi^a_I + i\xi_I \mathcal{B}^a_{\alpha A}\ , \tag{4.17}$$

and accordingly for the bosonic superfield connection $\nabla_{\alpha\beta}$. On all other fields, which are neutral under shifts generated by the $\mathcal{T}_k$, the action of $\nabla_{\alpha A}$ and $\hat{\nabla}_{\alpha A}$ coincides. The explicit form of the gauge transformations (2.23) is then given by

$$\begin{aligned} &\delta\Phi^I = \Lambda \cdot \Phi^I + \xi^I C\ , \quad \delta\Psi_{\alpha\dot{A}} = \Lambda \cdot \Psi_{\alpha\dot{A}}\,, \\ &\delta\hat{\mathcal{A}}_{\alpha A} = -\hat{\nabla}_{\alpha A} \cdot \Lambda\,, \qquad \delta\mathcal{B}_{\alpha A} = i\hat{\nabla}_{\alpha A} C + \Lambda \cdot \mathcal{B}_{\alpha A}\,, \end{aligned} \tag{4.18}$$

and analogous transformations for the bosonic superfields $\hat{\mathcal{A}}_{\alpha\beta}$, $\mathcal{B}_{\alpha\beta}$. These transformations lead to a homogeneous covariant transformation of the super covariant derivatives of Φ^I, i.e.

$$\delta(\nabla_{\alpha A} \Phi^I) = \Lambda \cdot (\nabla_{\alpha A} \Phi^I)\ , \tag{4.19}$$

which thus is neutral under local shifts in Φ^I. As for the covariant derivatives, also the conventional field strengths acquire extra terms only in the case of their action on the scalar superfields Φ^I. With the definitions (4.13), (4.15) one obtains for the anti-commutator

$$\begin{aligned}\{\nabla_{\alpha A}, \nabla_{\beta B}\} \cdot \Phi^I &= (2i\delta_{AB}\partial_{\alpha\beta} + D_{\alpha A}\mathcal{A}_{\beta B} + D_{\beta B}\mathcal{A}_{\alpha A} + \{\mathcal{A}_{\alpha A}, \mathcal{A}_{\beta B}\}) \cdot \Phi^I \\ &\equiv 2i\delta_{AB}\nabla_{\alpha\beta}\Phi^I + \hat{\mathcal{F}}_{\alpha A,\beta B} \cdot \Phi^I + \xi^I \mathcal{H}_{\alpha A,\beta B} \ , \end{aligned} \tag{4.20}$$

with the split of field strength into $\mathcal{F}_{\alpha A,\beta B} = \hat{\mathcal{F}}^a_{\alpha A,\beta B} T_a + \mathcal{H}^a_{\alpha A,\beta B}\mathcal{T}_a$, i.e.

$$\begin{aligned}\hat{\mathcal{F}}_{\alpha A,\beta B} &= D_{\alpha A}\hat{\mathcal{A}}_{\beta B} + D_{\beta B}\hat{\mathcal{A}}_{\alpha A} + \{\hat{\mathcal{A}}_{\alpha A}, \hat{\mathcal{A}}_{\beta B}\} - 2i\delta_{AB}\hat{\mathcal{A}}_{\alpha\beta} \ , \\ \mathcal{H}_{\alpha A,\beta B} &= \hat{\nabla}_{\alpha A}\mathcal{B}_{\beta B} + \hat{\nabla}_{\beta B}\mathcal{B}_{\alpha A} - 2i\delta_{AB}\mathcal{B}_{\alpha\beta} \ . \end{aligned} \tag{4.21}$$

Similarly, we split the bosonic field strength $\mathcal{F}_{\alpha\beta,\gamma\delta}$ into a part $\hat{\mathcal{F}}_{\alpha\beta,\gamma\delta}$ corresponding to the standard non-abelian Yang-Mills field strength of the gauge field $\hat{\mathcal{A}}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta,\gamma\delta} = \hat{\nabla}_{\alpha\beta}\mathcal{B}_{\gamma\delta} - \hat{\nabla}_{\gamma\delta}\mathcal{B}_{\alpha\beta}$ such that

$$\mathcal{F}_{\alpha\beta,\gamma\delta} \cdot \Phi^I = \hat{\mathcal{F}}_{\alpha\beta,\gamma\delta} \cdot \Phi^I + \xi^I \mathcal{H}_{\alpha\beta,\gamma\delta} \ . \tag{4.22}$$

It remains to find a solution for the tensor W_{IJ} living in the algebra (4.15) that satisfies the constraints (4.2) and (4.3). Our proposal is the following

$$W_{IJ} = W^{(0)a}_{IJ}\, T_a + W^{(1)a}_{IJ}\, \mathcal{T}_a \equiv -2\Phi^{a[I}\xi^{J]}\, T_a + f_{bc}{}^a \Phi^{Ib}\Phi^{Jc}\, \mathcal{T}_a \ , \tag{4.23}$$

where the superscripts $^{(0),(1)}$ refer to the $G_{\rm YM}$-covariant grading of the algebra (4.15). It is straightforward to verify that this function satisfies the constraint (4.2) as a consequence of (4.1). The argument is as in the last section: it follows with (4.1) that $\nabla_{\alpha A} W_{BC}$ w.r.t. $SO(8)$ transforms in the tensor product $\mathbf{8_v} \otimes \mathbf{8_c} = \mathbf{8_s} + \mathbf{56_s}$, which does not contain a $\mathbf{160_s}$. Moreover, with (4.14) one checks that

$$\begin{aligned}(W_{IJ} \cdot \Phi_K)^a &= W^{(0)b}_{IJ} f_{bc}{}^a \Phi^c_K + W^{(1)a}_{IJ}\, \xi_K \\ &= 3 f_{bc}{}^a \Phi^b_{[I}\Phi^c_J \xi_{K]} = (W_{[IJ} \cdot \Phi_{K]})^a \ , \end{aligned} \tag{4.24}$$

is completely antisymmetric in $[IJK]$, i.e. transforms in the $\mathbf{56_v}$, and thus also satisfies the constraint (4.3). This fixes the relative factor in (4.23). Finally, gauge covariance of the ansatz (4.23) requires

$$f_{[ab}{}^d\, f_{c]d}{}^e = 0 \ , \tag{4.25}$$

the standard Jacobi identities for the structure constants of $G_{\rm YM}$. To complete the construction, we evaluate the general formulae of section 3.2 for the solution (4.23). From (3.19) we obtain

$$\begin{aligned}\lambda_{\alpha A} &= -i\Gamma^A_{I\dot{A}} \left(\xi^I \Psi^a_{\alpha\dot{A}}\, T_a - i f_{bc}{}^a\, \Psi^b_{\alpha\dot{A}} \Phi^{Ic}\, \mathcal{T}_a\right) \ , \\ \rho_{\alpha\, ABC} &= \tfrac{1}{2} i\Gamma^{ABC}_{I\dot{A}} \left(\xi^I \Psi^a_{\alpha\dot{A}}\, T_a - i f_{bc}{}^a\, \Psi^b_{\alpha\dot{A}} \Phi^{Ic}\, \mathcal{T}_a\right) \ , \end{aligned} \tag{4.26}$$

and

$$V_{AB} = -\tfrac{3}{4} f_{bc}{}^{a} \Gamma^{KL}_{AB}\, \xi^{I} \xi_{[I} \Phi^{b}_{K} \Phi^{c}_{L]}\, T_{a} \\ \qquad + \Big(\tfrac{3}{4} f_{bc}{}^{a} f_{de}{}^{b} \Gamma^{KL}_{AB}\, \xi_{[I} \Phi^{d}_{K} \Phi^{e}_{L]} \Phi^{I\,c} - \tfrac{1}{2} i \varepsilon^{\alpha\beta} f_{bc}{}^{a}\, \Gamma^{AB}_{\dot A \dot B}\, \Psi^{b}_{\alpha \dot A} \Psi^{c}_{\beta \dot B} \Big) \mathcal{T}_{a} \; . \tag{4.27}$$

The first order CS-equations of motion (3.9) yields

$$\mathcal{F}_{\alpha\beta} = -\xi^{I}\, \nabla_{\alpha\beta} \Phi^{Ia}\, T_{a} - f_{bc}{}^{a} \left(\Phi^{Ib} \nabla_{\alpha\beta} \Phi^{Ic} + i \Psi^{b}_{\alpha \dot A} \Psi^{c}_{\beta \dot A} \right) \mathcal{T}_{a} \; . \tag{4.28}$$

Finally, the bosonic equations of motion (3.17) reduce to

$$\nabla^{2}\, \Phi^{a}_{I} = \tfrac{i}{2}\, \varepsilon^{\alpha\beta} \Gamma^{IJ}_{\dot A \dot B}\, f_{bc}{}^{a}\, \xi^{J} \Psi^{b}_{\alpha \dot A} \Psi^{c}_{\beta \dot B} + \tfrac{3}{2}\, f_{bc}{}^{d} f_{de}{}^{a} \xi^{N} \xi_{[N} \Phi^{b}_{I} \Phi^{c}_{J]} \Phi^{e}_{J} \; . \tag{4.29}$$

In order to show the equivalence to the standard formulation of three-dimensional $\mathcal{N} = 8$ Yang-Mills theories, one uses part of equations (4.28) to integrate out the vector field $\mathcal{B}_{\alpha\beta}$. Explicitly, we split the $SO(8)$ index $I \rightarrow (i, 8)$ with $i = 1, \ldots, 7$, set $\xi^{I} = \delta^{I8}$, and fix the gauge freedom $\delta\Phi^{I} = \xi^{I} C$ by setting $\Phi^{8} = 0$. Note that this gauge differs from the "transverse" gauge (2.29), which imposes also $\theta^{\alpha A} \mathcal{B}_{\alpha A} = 0$ and was used in the previous analysis to construct the superfield expansion.

Using (4.17) the T_a component of equation (4.28) reduces in this new gauge to

$$\hat{\mathcal{F}}_{\alpha\beta} = -\mathcal{B}_{\alpha\beta} \; , \tag{4.30}$$

and can be used to eliminate the gauge field $\mathcal{B}_{\alpha\beta}$ from all equations. In particular, using (4.22) the remaining component of equation (4.28) takes the form

$$\varepsilon^{\gamma\delta} \hat{\nabla}_{\gamma(\alpha} \hat{\mathcal{F}}_{\beta)\delta} = \tfrac{1}{2} [\Phi^{i}, \hat{\nabla}_{\alpha\beta} \Phi^{i}] + i \Psi^{\dot A}_{(\alpha} \Psi^{\dot A}_{\beta)} \; , \tag{4.31}$$

in which we recognize the standard second-order Yang-Mills equations of motion for the remaining gauge field $\hat{\mathcal{A}}_{\alpha\beta}$. The scalar equations of motion are obtained from (4.29) after imposing $\xi^{I} \Phi_{I} = 0$, and exhibit the quartic potential in the scalar fields Φ^{i}.

It is instructive to study this redualization of the three-dimensional degrees of freedom on a more fundamental level directly in terms of the superfield constraints. Upon setting $\Phi^{8} = 0$, the scalar constraint (4.1), or explicitly (3.12), implies that

$$i \mathcal{B}^{a}_{A\alpha} = \nabla_{A\alpha} \Phi^{8\,a} \;=\; i \Gamma^{8}_{A \dot A}\, \Psi^{\dot A\, a}_{\alpha} \; , \tag{4.32}$$

i.e. the vector superfield $\mathcal{B}_{A\alpha}$ which gauges the translations is identified with the fermion superfield $\Psi^{\dot A}_{\alpha}$. With (3.16) we thus obtain from (4.21)

$$\begin{aligned} \mathcal{H}_{A\alpha, B\beta} &= \mathcal{D}_{A\alpha} \mathcal{B}_{B\beta} + \mathcal{D}_{B\beta} \mathcal{B}_{A\alpha} - 2i \delta_{AB} \mathcal{B}_{\alpha\beta} \\ &= 2i \delta_{AB} (\nabla_{\alpha\beta} \Phi^{8\,a} - \mathcal{B}^{a}_{\alpha\beta})\, \mathcal{T}_{a} + \tfrac{1}{6} i \varepsilon_{\alpha\beta} (\Gamma^{IJK} \Gamma^{8})_{[AB]}\, (W_{IJ} \cdot \Phi_{K})^{a}\, \mathcal{T}_{a} \\ &= \tfrac{1}{2} i \varepsilon_{\alpha\beta} (\Gamma^{IJK} \Gamma^{8})_{[AB]}\, f_{bc}{}^{a}\, \Phi^{b}_{[I} \Phi^{c}_{J}\, \xi_{K]}\, \mathcal{T}_{a} \\ &= 2i\, \varepsilon_{\alpha\beta}\, W^{(1)\,a}_{AB}\, \mathcal{T}_{a} \; . \end{aligned} \tag{4.33}$$

I.e. the constraint (3.1) is automatically satisfied for the $\mathcal{T}_a$ component of the superfield strength. The remaining part of this superfield constraint yields

$$\hat{\mathcal{F}}_{A\alpha,B\beta} = \tfrac{1}{2} i\varepsilon_{\alpha\beta}\Gamma^{IJ}_{AB}\, W^{(0)a}_{IJ}\, T_a \;=\; i\varepsilon_{\alpha\beta}\Gamma^{8i}_{AB}\,\Phi^i \;, \tag{4.34}$$

or equivalently

$$\{\hat{\nabla}_{A\alpha}, \hat{\nabla}_{B\beta}\} = 2i\delta_{AB}\hat{\nabla}_{\alpha\beta} + i\varepsilon_{\alpha\beta}\Gamma^{8i}_{AB}\,\Phi^i \;. \tag{4.35}$$

If we take this equation as a definition for the scalar fields Φ^i, the Bianchi identities for (4.35) induce the matter superfield constraint (3.12). In this respect, equation (4.35) may thus be considered as a weaker version of the constraint (2.25), which accordingly gives rise to Yang-Mills dynamics rather than to a Chern-Simons dynamics for the gauge fields involved. Moreover, we recognize in (4.35) the remnant of the superfield constraint underlying ten-dimensional super Yang-Mills theory [19, 25]

$$\{\nabla_{\mathcal{A}}, \nabla_{\mathcal{B}}\} = 2i\,\Gamma^{\mathcal{I}}_{\mathcal{AB}}\,\nabla_{\mathcal{I}} \;, \tag{4.36}$$

with $SO(9,1)$ vector and spinor indices $\mathcal{I}$ and $\mathcal{A}$, respectively, after breaking the Lorentz group $SO(9,1) \to SO(2,1)\times SO(7)$ and truncating the partial derivatives w.r.t. the seven internal coordinates. The scalar fields Φ^i represent the seven internal components of the ten-dimensional vector.

5. Conclusions and outlook

In this paper, we have given a systematic analysis of the $\mathcal{N}=8$ superspace constraints in three space-time dimensions. The general coupling between vector and scalar supermultiplets is encoded in the deformation potential W_{AB} which is a function of the matter fields subject to the W-constraints (4.2) and (4.3). The full equations of motion are given by equations (3.9) and (3.17). We have given the two solutions (4.4) and (4.23) to these constraints including the conformal BLG model and to three-dimensional Yang-Mills theory, respectively. The presented results and the universal formalism in which all known $\mathcal{N}=8$ three-dimensional theories have been embedded suggest a number of possible generalizations and directions of further research of which we list a few in the following.

- In the course of this paper we have met and analyzed various different constraints for the super field strength $\mathcal{F}_{\alpha A,\beta B}$. In its strongest version (2.25) the field strength $\mathcal{F}_{\alpha A,\beta B}$ is set to zero which gives rise to a (first order) Chern-Simons dynamics of the bosonic gauge field. A weaker version of the constraint is (3.2) which allows for a non-vanishing part in the irreducible $(\mathbf{1},\mathbf{28})$. As shown in appendix A, this leads to an enlarged vector multiplet with essentially no dynamics (apart from certain first order constraint equations on the higher order components of the multiplet). Yet another version of the constraint has been encountered in (4.35) upon breaking $SO(8)$ down to $SO(7)$ and allowing an irreducible $(\mathbf{1}, \mathbf{7})$ in the super field strength. As discussed above, this is related to a ten-dimensional origin of the theory and induces a (second order) Yang-Mills dynamics for the bosonic gauge field. In order of increasing constraints, these cases may be tabulated as

$$\begin{aligned}
\{\nabla_{A\alpha}, \nabla_{B\beta}\}\Big|_{(3,35)} &= 0 \quad \Longrightarrow \quad \text{no dynamics}\ , \\
\{\nabla_{A\alpha}, \nabla_{B\beta}\}\Big|_{(3,35)+(1,21)} &= 0 \quad \Longrightarrow \quad \text{Yang-Mills dynamics}\ , \\
\{\nabla_{A\alpha}, \nabla_{B\beta}\}\Big|_{(3,35)+(1,28)} &= 0 \quad \Longrightarrow \quad \text{free Chern-Simons dynamics}\ ,
\end{aligned} \tag{5.1}$$

and show how the field content and the dynamics becomes more restrictive as a function of the constraints. It would be very interesting to perform a similar analysis for other versions of the constraint upon breaking the original form under various subgroups of $SO(8)$ and to study the resulting multiplet structures, their dynamics and a possible higher-dimensional origin.

- As shown in appendix A, the first constraint in (5.1) admits the representation as a partial flatness condition for the integrability of an auxiliary linear system. No such representation is known for the constraint (3.1) with W_{AB} being a deformation potential, as we have studied it in this paper. However, as we have discussed in section 4.2, for the particular solution (4.23) of the W-constraints, the super field strength may be brought into the form (4.35) which descends from the zero-curvature condition on super null lines (4.36) of the linear system underlying the ten-dimensional Yang-Mills equations of motion [25]. Dimensional reduction does not guarantee the existence of a linear auxiliary system and a corresponding twistor space description. For example for the $\mathcal{N} = 4$ SYM theory in four dimensions no such system is known, only the $\mathcal{N} = 3$ superspace formulation has been described in these geometric terms so far [24]. However, the dimensional reduction of the the ten dimensional SYM superspace constraints to six dimensions, describing six-dimensional $\mathcal{N} = 2$ SYM, can be reformulated as a linear auxiliary system[12]. In three dimensions a twistorial description of SYM has been given in $\mathcal{N} = 6$ superspace [41]. However, it is an interesting question if there exists an auxiliary linear system and an associated twistor space description for the solution (4.4) of the W-constraints

[12] With $x^{ij} = x^{[ij]} = \frac{1}{2}\varepsilon^{ijkl}x_{kl}$ being a six-dimensional vector $(i, j = 1, \ldots, 4)$ one finds that the integrability conditions of $x^{ij}\nabla_{j\alpha}\mathcal{S} = x_{ij}\nabla^{j}_{\dot\alpha}\mathcal{S} = x^{ij}\nabla_{ij}\mathcal{S} = 0$ are equivalent to the superspace constraints for the six-dimensional $\mathcal{N} = 2$ SYM theory as given in [19], *iff* x^{ij} is a null vector. The geometry of these null-vectors and the corresponding twistor space were discussed in [40], it is natural to expect that there exists a twistor space formulation of the six-dimensional $\mathcal{N} = 2$ SYM theory.

which eventually would give rise to a linear system and associated twistor space formulation underlying the equations of motion of the conformal BLG model. The covariance of our formalism suggest a study of this question analogous to SYM theories.

- In this paper we have studied the interactions between scalar and vector superfields induced by a deformation (3.1) of the super field strength. A natural generalization of this ansatz would also include the remaining irreducible term

$$\{\nabla_{\alpha A}, \nabla_{\beta B}\} = 2i\left(\delta_{AB}\nabla_{\alpha\beta} + \varepsilon_{\alpha\beta}W_{AB} + J_{\alpha\beta\, AB}\right) \ , \tag{5.2}$$

with a tensor $J_{\alpha\beta\, AB} = J_{(\alpha\beta)(AB)}$, traceless in (AB), that is now likewise given as a function of the matter fields. An analysis similar to the one performed in the main text, shows that in presence of a non-vanishing $J_{\alpha\beta\, AB}$ the differential W-constraint (4.2) is modified to

$$\begin{aligned} \varepsilon^{\beta\gamma}\,\nabla_{A\beta}J_{\gamma\alpha\, BC}\Big|_{\mathbf{160_s}} &= \nabla_{\alpha A}W_{BC}\Big|_{\mathbf{160_s}} \ , \\ \nabla_{A(\alpha}J_{\beta\gamma)\, BC}\Big|_{\mathbf{112_s}} &= 0 \ , \end{aligned} \tag{5.3}$$

where the projectors on the l.h.s. refer to the irreducible parts of the tensor product $\mathbf{8_s} \otimes \mathbf{35_s} = \mathbf{8_s} \oplus \mathbf{112_s} \oplus \mathbf{160_s}$ in which $\nabla_{\alpha A}J_{\beta\gamma\, BC}$ transforms w.r.t. $SO(8)$. Likewise, upon coupling to scalar superfields, the algebraic W-constraint (4.3) is extended to

$$W_{AB}\cdot\Phi^I\Big|_{\mathbf{160_v}} = 0 \ = \ J_{\alpha\beta\, AB}\cdot\Phi^I\Big|_{\mathbf{224_v}} \ . \tag{5.4}$$

We expect that similar to the analysis presented in the text, these constraints will be sufficient to guarantee consistency of the system (5.2) coupled to scalar superfields. It remains an open question to find solutions of the extended set of constraints (5.3), (5.4) that would give rise to more general $\mathcal{N} = 8$ theories.

- Along similar lines, the system (4.1)–(4.3) can be generalized by deforming the matter superfield constraint (4.1), i.e. by allowing more general contributions

$$\nabla_{\alpha A}\,\Phi^I = \Gamma^I_{A\dot{A}}\Psi_{\alpha\dot{A}} + \Gamma^{\dot{A}\dot{B}\dot{C}}_{IA}\,\Theta_{\alpha\dot{A}\dot{B}\dot{C}} \ , \tag{5.5}$$

where now $\Theta_{\alpha\dot{A}\dot{B}\dot{C}}$ is considered as a function of the superfields Φ^I, $\Psi_{\alpha\dot{A}}$ (subject to a number of differential and algebraic constraints). A similar strategy has been used in [42] in order to constrain the higher order α' corrections to ten-dimensional super Yang-Mills theory. In the present context, a viable strategy in order to describe higher order corrections to the models may be to implement the algebraic W-constraint (4.3) by adequate choice of the deformation potential W_{AB} while solving the differential W-constraint (4.2) for this functional by suitably tuning the Θ contribution in (5.5) that modifies (4.1). In this context it is also possible to consider non-polynomial generalizations of the ansatz (4.4) which are scale invariant. The verification of the conformal symmetry of the resulting models can be conveniently carried out by representing the superconformal algebra on the $\mathcal{N}=8$ superspace. These steps represent a possibility for determing quantum corrections without relying on perturbation theory.

- The generic scalar field equations of motion (3.17) that we have derived as a consequence of the superspace constraints exhibit various terms containing the deformation potential W_{AB}, as well as the derived quantities $\lambda_{\alpha A}$, $\rho_{\alpha ABC}$ and V^{IJ}. However, when explicitly evaluating these terms for the explicit models in (4.10) and (4.29), we observe that all the terms give rise to only two distinct contributions to the equations of motion, a purely bosonic term and a single term bilinear in the fermions. This raises the question if this reduction of the general equation is related to some (yet undiscovered) underlying structure of the generic theory or if there exist more general solutions to the W-constraints (4.2), (4.3) for which the different terms of (3.17) do give contributions of different type. The question may be related to the fact that both our explicit solutions (4.4) and (4.23) satisfy an algebraic equation which is actually stronger than (4.3) and reads

$$W_{IJ} \cdot \Phi_K \Big|_{\mathbf{8_v}+\mathbf{160_v}} = 0 \,. \tag{5.6}$$

  It would be highly interesting to understand if (5.6) is a (hidden) consequence of the constraints (4.2), (4.3) or if the latter admit solutions with a non-trivial component in the $\mathbf{8_v}$. With regard to the supersymmetry transformations (3.25) this would also have an impact on the BPS equation of this system and thus generalize the original Basu-Harvey equation [43].

- Finally, it is a natural task to perform a similar analysis of superspace constraints for the theories with less supersymmetry. Of particular interest is the case $\mathcal{N}=6$, including the theories of [5, 44]. The relation to the harmonic superspace approach [45] and the pure spinor formulations [46] in this case remain to be investigated.

We hope to come back to some of these issues in future work.

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A. A weaker constraint

In this appendix, we complete the discussion of the constraint system (3.2), i.e. of a vector multiplet with W_{AB} considered as an independent field defined by (3.1). In this case, the constraint (3.1) can be understood as a partial flatness condition,

$$\mathcal{F}_{\alpha A,\beta B} + \mathcal{F}_{\alpha B,\beta A} = 0 \ , \tag{A.1}$$

and therefore admits an equivalent formulation as an linear auxiliary system,

$$\lambda^{\alpha\beta}\nabla_{A\beta}\,\mathcal{S}(\lambda) = 0 \ , \qquad \lambda^{\alpha\beta}\nabla_{\alpha\beta}\,\mathcal{S}(\lambda) \ = \ 0 \ , \tag{A.2}$$

with a light-like vector $\lambda^{\alpha\beta}\lambda_{\alpha\beta} = 0$, such that integrability of (A.2) implies (A.1). Light-like vectors in $\mathbb{R}^{1,2}$ are parametrized by TS^1, the Minkowski space version of the mini-twistor space [47], which suggest the existence of a corresponding twistor space formulation of this system.

To keep the analysis of the multiplet structure transparent we analyze the system (3.1) for abelian vector superfields, for which the resulting equations simplify considerably. The full non-abelian analysis does not add any conceptual challenges or modifications of the component field content except for the fact that all fields are matrices of the non abelian Lie algebra.

The conditions due to the Bianchi identities (3.4),(3.5), (3.7) and (3.8) are of the same form as in the non-abelian case, except that the covariant derivatives acting in the adjoint representation can be replaced by partial derivatives in the abelian case. The integrability conditions (3.11) are now genuine nontrivial conditions on the superfields. In the abelian case, they simplify considerably to

$$\begin{aligned} D_{\alpha A}\,\rho_{\beta BCD} &= 3i\partial_{\alpha\beta}W_{[BC}\delta_{D]A} - \tfrac{3i}{2}\varepsilon_{\alpha\beta}\delta_{A[B}V_{CD]} + iU_{\alpha\beta\,ABCD} \ , \\ D_{\alpha A}V_{BC} &= 2\varepsilon^{\beta\gamma}\partial_{\alpha\beta}\left(\delta_{A[B}\lambda_{C]\gamma} - \rho_{\gamma ABC}\right) \ , \\ D_{\alpha A}U_{\beta\gamma\,BCDE} &= 8\delta^{A[B}\partial_{\alpha(\beta}\rho_{\gamma)}^{CDE]} - 4\delta^{A[B}\partial_{\beta\gamma}\rho_{\alpha}^{CDE]} + \tau_{\alpha\beta\gamma\,ABCDE} \ . \end{aligned} \tag{A.3}$$

Evaluating the anti-commutator (3.1) on the last equation of (A.3) determines the superderivative of the tensor $\tau_{\alpha\beta\gamma\,ABCDE}$ as

$$\begin{aligned} D_{\alpha A}\tau_{\beta_1\beta_2\beta_3\,B_1\cdots B_5} = {} & 10i\delta^{A[B_1}\partial_{\alpha(\beta_1}U_{\beta_2\beta_3)}^{B_2\cdots B_5]} - 5i\delta^{A[B_1}\partial_{(\beta_1\beta_2}U_{\beta_3)\alpha}^{B_2\cdots B_5]} \\ & + iT_{\alpha\beta_1\beta_2\beta_3\,AB_1\cdots B_5} \ , \end{aligned} \tag{A.4}$$

up to a tensor $T_{\alpha_1\cdots\alpha_4\,A_1\cdots A_6} = T_{(\alpha_1\cdots\alpha_4)\,[A_1\cdots A_6]}$. Iterating this procedure, we finally arrive at the (closed) system

$$\begin{aligned}
D_{\alpha A} T_{\beta_1\cdots\beta_4\, B_1\cdots B_6} &= 12\, \delta^{A[B_1} \partial_{\alpha(\beta_1} T^{B_2\cdots B_6]}_{\beta_2\beta_3\beta_4)} - 6\, \delta^{A[B_1} \partial_{(\beta_1\beta_2} T^{B_2\cdots B_6]}_{\beta_3\beta_4)\alpha} \\
&\quad + \sigma_{\alpha\beta_1\cdots\beta_4\, AB_1\cdots B_6} \ , \\[2mm]
D_{\alpha A} \sigma_{\beta_1\cdots\beta_5\, B_1\cdots B_7} &= 14 i \delta^{A[B_1} \partial_{\alpha(\beta_1} T^{B_2\cdots B_7]}_{\beta_2\cdots\beta_5)} - 7 i \delta^{A[B_1} \partial_{(\beta_1\beta_2} T^{B_2\cdots B_7]}_{\beta_3\beta_4\beta_5)\alpha} \\
&\quad + i S_{\alpha\beta_1\cdots\beta_5\, AB_1\cdots B_7} \ , \\[2mm]
D_{\alpha A} S_{\beta_1\cdots\beta_6\, B_1\cdots B_8} &= 16\, \delta^{A[B_1} \partial_{\alpha(\beta_1} \sigma^{B_2\cdots B_8]}_{\beta_2\cdots\beta_6)} - 8\, \delta^{A[B_1} \partial_{(\beta_1\beta_2} \sigma^{B_2\cdots B_8]}_{\beta_3\cdots\beta_6)\alpha} \ . \qquad \text{(A.5)}
\end{aligned}$$

with additional tensors σ and S, which are completely symmetric (antisymmetric) in their $SO(2,1)$ ($SO(8)$) indices. Evaluating the anti-commutator (3.1) on the first equation of (A.3) leads to two consistency equations for the tensor and $U_{ABCD\,\alpha\beta}$ and the fourth (abelian) Bianchi identity:

$$\partial^{\alpha\beta} \mathcal{F}_{\alpha\beta} = 0 \ , \qquad \partial^{\alpha\beta} U_{\alpha\beta\, ABCD} \ = \ 0 \ , \qquad \text{(A.6)}$$

Similarly, consistency of (A.4), (A.5) requires the first order equations

$$\partial^{\alpha\beta} \tau_{\alpha\beta\gamma\, A_1\cdots A_5} \ = \ 0 \ , \qquad \partial^{\alpha\beta} T_{\alpha\beta\gamma_1\gamma_2\, A_1\cdots A_6} \ = \ 0 \ , \qquad \text{(A.7)}$$

and analogous equations for σ and S, showing that in the abelian case these tensors are conserved higher spin currents. In the non-abelian case, a crucial modification takes place. First, partial derivatives are replaced by covariant derivatives and second, the r.h.s. of the equations (A.6), (A.7) (except for the Bianchi identity) receive non-vanishing contributions from commutators of the non-abelian fields.

Superfield expansion, multiplet structure

The obtained closed system of superderivatives of superfields (3.1), (3.4), (3.5), (3.7), (3.8) and (A.3), (A.4), (A.5) allows to define a closed recursive system to systematically obtain the expansion in terms of component fields. Contracting all these equations with $\theta^{\alpha A}$ gives

$$\begin{aligned}
(1+\mathcal{R})\, \mathcal{A}_{\alpha A} &= 2i\theta^{\beta A} \mathcal{A}_{\alpha\beta} + 2i \varepsilon_{\alpha\beta} \theta^{\beta B} W_{AB} \ , \\
\mathcal{R}\, \mathcal{A}_{\alpha\beta} &= \theta^{\gamma A} \varepsilon_{\gamma(\alpha} \lambda_{\beta)A} \ , \\
\mathcal{R}\, W_{AB} &= \theta^{\delta D} (\delta_{D[A} \lambda_{B]\delta} + \rho_{\delta DAB}) \ , \\
\mathcal{R}\, \lambda_{\alpha A} &= i\theta^{\delta D} (\delta_{DA} \mathcal{F}_{\delta\alpha} + 2\, \partial_{\delta\alpha} W_{DA} + \varepsilon_{\delta\alpha} V_{DA}) \ , \\
&\cdots \\
\mathcal{R}\, S_{\alpha_1\cdots\alpha_6\, A_1\cdots A_8} &= 16\, \theta^{\beta[A_1} \partial_{\beta(\alpha_1} \sigma^{A_2\cdots A_8]}_{\alpha_2\cdots\alpha_6)} - 8\, \theta^{\beta[A_1} \partial_{(\alpha_1\alpha_2} \sigma^{A_2\cdots A_8]}_{\alpha_3\cdots\alpha_6)\beta} \ , \qquad \text{(A.8)}
\end{aligned}$$

generalizing (2.30). This shows that the superfield $\mathcal{A}_{\alpha A}$ is entirely determined in terms of the lowest components of all the superfields involved

$$\begin{aligned}
\mathcal{A}_{\beta B} &= i(\theta^{\alpha}_{B}\ A_{\alpha\beta} + \theta^{\alpha A} \varepsilon_{\alpha\beta} \overset{\circ}{W}_{AB}) \\
&\quad + \tfrac{2i}{3}\, \theta^{\alpha A} \theta^{\gamma C} (\delta_{AB}\ \varepsilon_{\gamma(\alpha} \overset{\circ}{\lambda}_{\beta)C} + \varepsilon_{\alpha\beta} \delta_{C[A} \overset{\circ}{\lambda}_{B]\gamma} + \varepsilon_{\alpha\beta}\ \overset{\circ}{\rho}_{\gamma CAB}) + \ldots \ . \qquad \text{(A.9)}
\end{aligned}$$

θ^N	Field	Representation under $SO(2,1)\times SO(8)$
0	—	—
1	$A_{\alpha\beta}+\overset{\circ}{W}_{AB}$	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{28})$
2	$\overset{\circ}{\lambda}_{\alpha A}+\overset{\circ}{\rho}_{\alpha\, ABC}$	$(\mathbf{2},\mathbf{8_s})+(\mathbf{2},\mathbf{56_s})$
3	$\overset{\circ}{V}_{AB}+\overset{\circ}{U}_{\alpha\beta\, ABCD}$	$(\mathbf{1},\mathbf{28})+(\mathbf{3}-\mathbf{1},\mathbf{35_v}+\mathbf{35_c})$
4	$\overset{\circ}{\tau}_{\alpha\beta\gamma\, ABCDE}$	$(\mathbf{4}-\mathbf{2},\mathbf{56_s})$
5	$\overset{\circ}{T}_{\alpha_1\ldots\alpha_4\, A_1\ldots A_6}$	$(\mathbf{5}-\mathbf{3},\mathbf{28})$
6	$\overset{\circ}{\sigma}_{\alpha_1\ldots\alpha_5\, A_1\ldots A_7}$	$(\mathbf{6}-\mathbf{4},\mathbf{8_s})$
7	$\overset{\circ}{S}_{\alpha_1\ldots\alpha_6\, A_1\ldots A_8}$	$(\mathbf{7}-\mathbf{5},\mathbf{1})$
8	—	—

Table 1: Superfield expansion of the vector field $\mathcal{A}_{\alpha A}$ induced by the weaker constraint (3.2). The negative multiplicities of representations w.r.t. $SO(2,1)$ correspond to the first order constraint equations which these fields satisfy.

The only equations that these fields must obey are the first order constraint equations (A.6), (A.7), etc. The superfield expansion of $\mathcal{A}_{A\alpha}$ is summarized in table 1, where the negative multiplicities refer to the first order constraint equations. The resulting multiplet is thus neither on-shell (as there are genuine field equations for its components) nor entirely off-shell (due to the presence of the constraint equations). Counting the field content of table 1 reveals 257 bosonic + 256 fermionic degrees of freedom with the extra bosonic singlet corresponding to the gauge freedom of the vector field $A_{\alpha\beta}$. Interestingly, the same multiplet has appeared in [48] in the context of reducing the superspace constraints of ten-dimensional Yang-Mills theories down to seven dimensions.

The relation between $F_{\alpha\beta}$ and $A_{\alpha\beta}$ may give an idea how to resolve the constrained fields in terms of genuine off-shell fields. E.g. in the abelian theory, the 70 conserved currents $U_{ABCD\,\alpha\beta}$ can be written in the form

$$U^{ABCD}{}_{\alpha\beta} = \varepsilon^{\gamma\delta}\partial_{\gamma(\alpha}B^{ABCD}{}_{\beta)\delta}\ , \tag{A.10}$$

as the field strengths of 70 off-shell and unconstrained vector fields $B^{ABCD}{}_{\alpha\beta}$. For the higher spin fields in contrast, this is less clear. In particular, the non-abelian generalization upon which the components $U^{ABCD}{}_{\alpha\beta}$, $\tau_{\alpha\beta\gamma\, A_1\cdots A_5}$, etc., are no longer covariantly conserved currents, makes it even harder to see if there exists an formulation in terms of genuine off-shell fields.

In the non-abelian case the superfield expansion of $\mathcal{A}_{\alpha A}$ to second order in θ is formally the same as in (A.9). For the basic matter superfields and the bosonic gauge superfield one finds to second order in θ:

$$
\begin{aligned}
\Phi^I &= \phi^I + i\theta^{\alpha A}\Gamma^I_{A\dot{A}}\psi_{\alpha\dot{A}} + \tfrac{i}{2}\,\theta^{\alpha A}\theta^{\beta B}\;\Gamma^{IJ}_{AB}\;\overset{\circ}{\nabla}_{\alpha\beta}\phi^J \\
&\qquad -\tfrac{i}{4}\,\theta^{\alpha A}\theta^{\beta B}\varepsilon_{\alpha\beta}(\tfrac{1}{7}\,\delta_{AB}\;\overset{\circ}{W}_{IJ}\cdot\phi^J - \tfrac{1}{6}\,\Gamma^{ILMN}_{AB}\;\overset{\circ}{W}_{LM}\cdot\phi_N) + \dots\ , \\
\Psi_{\beta\dot{A}} &= \psi_{\beta\dot{A}} + \theta^{\alpha A}(\Gamma^I_{A\dot{A}}\overset{\circ}{\nabla}_{\alpha\beta}\phi^I + \tfrac{1}{2}\;\varepsilon_{\alpha\beta}P^{IJK}_{A\dot{A}}\;\overset{\circ}{W}_{IJ}\cdot\phi_K) \\
&\qquad +\tfrac{i}{2}\;\theta^{\alpha A}\theta^{\gamma C}\left(\Gamma^I_{A\dot{A}}\Gamma^I_{C\dot{C}}\overset{\circ}{\nabla}_{\alpha\beta}\;\psi_{\gamma\dot{C}} + P^{IJK}_{A\dot{A}}\Gamma^K_{C\dot{C}}\;\overset{\circ}{W}_{IJ}\cdot\psi_{\gamma\dot{C}}\right) \\
&\qquad +\tfrac{1}{2}\,\theta^{\alpha A}\theta^{\gamma C}\left(\Gamma^K_{A\dot{A}}\;\varepsilon_{\gamma(\alpha}\overset{\circ}{\lambda}_{\beta)C}\cdot\phi^K + \tfrac{1}{4}\,\Gamma^{IJ}_{BD}P^{IJK}_{A\dot{A}}(\delta_{C[B}\overset{\circ}{\lambda}_{D]\gamma} + \overset{\circ}{\rho}_{\gamma CBD})\cdot\phi_K\right) + \dots\ , \\
\mathcal{A}_{\alpha\beta} &= A_{\alpha\beta} + \theta^{\gamma C}\varepsilon_{\gamma(\alpha}\overset{\circ}{\lambda}_{\beta)C} \\
&\qquad +\tfrac{i}{2}\,\theta^{\gamma C}\theta^{\delta D}[\tfrac{1}{2}\varepsilon_{\gamma\delta}\;\delta_{CD}F_{\alpha\beta} - 2\varepsilon_{\gamma(\alpha}\overset{\circ}{\nabla}_{\beta)\delta}\overset{\circ}{W}_{CD} + \varepsilon_{\gamma(\alpha}\varepsilon_{\beta)\delta}\overset{\circ}{V}_{CD}] + \dots\ ,
\end{aligned}
\tag{A.11}
$$

where we have introduced the abbreviation $P^{IJK}_{A\dot{A}} = \frac{1}{7}\,\Gamma^I_{A\dot{A}}\delta^{JK} + \frac{1}{6}\,\Gamma^{IJK}_{A\dot{A}}$.

B. $SO(8)$ relations

The group $SO(8)$ (we consider mainly the associated Lie-algebra $so(8)$ and we are somewhat cavalier regarding the difference) has rather special properties. It admits a Majorana-Weyl representation in terms of real eight-component Spinors and the chirally conjugated ones, and consequently there are three inequivalent (real) eight dimensional irreducible representations $\mathbf{8_s}$, $\mathbf{8_c}$ and $\mathbf{8_v}$, where $\mathbf{8_v}$ is the vector representation of $SO(8)$. The source of this "accidental" coincidence in the dimensionality is the underlying triality symmetry which can be seen from the associated Dynkin diagram.

A commonly chosen Majorana-Weyl representation of the $SO(8)$ Gamma matrices $\tilde{\Gamma}^I$ is given in terms of real 8×8 blocks:

$$
\tilde{\Gamma}^I = \begin{bmatrix} 0 & \Gamma^I \\ \bar{\Gamma}^I & 0 \end{bmatrix}\ , \tag{B.1}
$$

where $\bar{\Gamma}^I = (\Gamma^I)^T$. We denote the components of the matrices Γ^I by

$$
\Gamma^I_{A\dot{B}} \qquad \text{with} \qquad I, A, \dot{B} = 1, \dots, 8\ \ , \tag{B.2}
$$

and we do not introduce a separate symbol for the transposed matrices $\bar{\Gamma}^I$, which in fact occur only in this appendix to keep the notation more compact. The basic algebraic relations for these matrices are[13]

$$
\Gamma^{(I}\bar{\Gamma}^{J)} = \bar{\Gamma}^{(I}\Gamma^{J)} = \delta^{IJ}\;\mathbb{1}_8\ \ , \tag{B.3}
$$

[13] We denote symmetrization/antisymmetrization in indices by () and [], respectively, and (anti)-symmetrizations are always defined with weight one.

and an explicit representation of these matrices can be found for example in [49]. Further we introduce the totally antisymmetrized products

$$\begin{aligned} \Gamma^{I_1 I_2 \ldots I_n}_{AB} &:= (\Gamma^{[I_1}\bar\Gamma^{I_2}\ldots\bar\Gamma^{I_n]})_{AB} \quad \ldots \quad n \text{ even} , \\ \Gamma^{I_1 I_2 \ldots I_n}_{A\dot A} &:= (\Gamma^{[I_1}\bar\Gamma^{I_2}\ldots\Gamma^{I_n]})_{A\dot A} \quad \ldots \quad n \text{ odd} , \end{aligned} \tag{B.4}$$

and analogously one can define matrices $\bar\Gamma^{IJK\ldots}$ where the alternating sequence of matrix products starts with a the transpose matrix $\bar\Gamma^I$, replacing dotted and undotted indices in (B.4). These matrices have the following symmetry properties under transposition:

$$\begin{aligned} \Gamma^{I_1 I_2 \ldots I_n}_{AB} &= (-)^{n(n-1)/2}\ \Gamma^{I_1 I_2 \ldots I_n}_{BA} \quad \ldots \quad n \text{ even} , \\ (\Gamma^{I_1 I_2 \ldots I_n})^T &= (-)^{n(n-1)/2}\ \bar\Gamma^{I_1 I_2 \ldots I_n} \quad \ldots \quad n \text{ odd} . \end{aligned} \tag{B.5}$$

Identities. We give here a number of useful Γ-matrix identities which where used in the calculations of the main text. We first give a basic identity, which is also the origin of the triality relations that we used in this work (see below):

$$\Gamma^I_{A\dot A}\Gamma^I_{B\dot B} + \Gamma^I_{A\dot B}\Gamma^I_{B\dot A} = 2\ \delta_{AB}\delta_{\dot A\dot B} \quad . \tag{B.6}$$

Defining $\delta^{I_1\ldots I_n}_{J_1\ldots J_n} := \delta^{I_1}_{[J_1}\ldots\delta^{I_n}_{J_n]}$ we have the following identities:

- Traces

$$\begin{aligned} \mathrm{Tr}[\Gamma^{I_1\ldots I_n}] &= 0 \quad \text{for} \quad n > 1 , \\ \mathrm{Tr}[\Gamma^{IJ}\Gamma^{KL}] &= -16\,\delta^{IJ}_{KL} , \\ \mathrm{Tr}[\Gamma^{IJK}\Gamma^{LMN}] &= 48\,\delta^{IJK}_{LMN} , \\ \mathrm{Tr}[\Gamma^{IJKL}\Gamma^{MNOP}] &= 8\,(24\,\delta^{IJKL}_{MNOP} + \varepsilon^{IJKLMNOP}) , \\ \mathrm{Tr}[\Gamma^{IL}\Gamma^{JM}\Gamma^{KN}] &= 32\,(\delta^{[J}_{L}\delta^{M]I}_{KN} - \delta^{[J}_{I}\delta^{M]L}_{KN}) . \end{aligned} \tag{B.7}$$

- Products

$$\begin{aligned} (\Gamma^{IJ}\Gamma^{KL})_{AB} &= \Gamma^{IJKL} - 2\,(\delta^{K[I}\Gamma^{J]L}_{AB} - \delta^{L[I}\Gamma^{J]K}_{AB}) - 2\,\delta_{AB}\ \delta^{IJ}_{KL} , \\ (\Gamma^{LN}\Gamma^{IJKN})_{AB} &= 4\ \Gamma^{IJKL}_{AB} - 30\,\Gamma^{[IJ}_{AB}\delta^{K]L} , \\ (\bar\Gamma^{ILN}\Gamma^{IJK})_{\dot A\dot B} &= 4\,\bar\Gamma^{LNJK}_{\dot A\dot B} + 10\,(\bar\Gamma^{L[K}_{\dot A\dot B}\delta^{J]N} - \bar\Gamma^{N[K}_{\dot A\dot B}\delta^{J]L}) + 12\,\delta^{LN}_{KJ}\delta_{\dot A\dot B} . \end{aligned} \tag{B.8}$$

- Tensor products

$$
\begin{aligned}
\Gamma^{IJ}_{AB}\Gamma^{IJ}_{CD} &= 16\,\delta^{AB}_{CD}\,,\\
\Gamma^{IJ}_{AB}\bar\Gamma^{IJ}_{\dot C\dot D} &= 2\,\Gamma^{I}_{A\dot C}\Gamma^{I}_{B\dot D} - 2\,\Gamma^{I}_{B\dot C}\Gamma^{I}_{A\dot D}\,,\\
\Gamma^{IJKL}_{AB}\Gamma^{L}_{C\dot C} &= -\delta_{AB}\Gamma^{IJK}_{C\dot C} + 2\,\delta_{C(A}\Gamma^{IJK}_{B)\dot C} + 6\,\Gamma^{[IJ}_{C(A}\Gamma^{K]}_{B)\dot C}\,,\\
\Gamma^{IJ}_{AB}\Gamma^{IJK}_{C\dot C} &= -2\,\Gamma^{I}_{C\dot C}\Gamma^{IK}_{AB} + 16\,\delta_{C[A}\Gamma^{K}_{B]\dot C}\,,\\
(\bar\Gamma^{J}\Gamma^{I}\bar\Gamma^{K})_{\dot A A}\Gamma^{JK}_{BC} &= 16\,\delta_{A[B}\Gamma^{I}_{C]\dot A} - 2\,\Gamma^{J}_{A\dot A}\Gamma^{JI}_{BC}\,,\\
\Gamma^{IJ}_{AB}\Gamma^{IJKL}_{CD} &= 2\,\delta_{CD}\Gamma^{KL}_{AB} - 8\,(\delta_{A(C}\Gamma^{KL}_{D)B} - \delta_{B(C}\Gamma^{KL}_{D)A})\,,\\
\Gamma^{IJK}_{A\dot A}\Gamma^{IJK}_{B\dot B} &= 48\,\delta_{AB}\delta_{\dot A\dot B} - 6\,\Gamma^{I}_{A\dot A}\Gamma^{I}_{B\dot B}\,,\\
\Gamma^{IJK}_{A\dot A}\Gamma^{IJKL}_{BC} &= 48\,\delta_{A(B}\Gamma^{L}_{C)\dot A} - 6\,\delta_{BC}\Gamma^{L}_{A\dot A}\,.
\end{aligned}
\tag{B.9}
$$

Triality. Here we explain some triality relations which were used in the main text. The basic identity for these considerations is equation (B.6) which is exactly the same relation as (B.3) if we consider "new" matrices[14] Γ^A with matrix components $\Gamma^A_{I\dot B} := \Gamma^I_{A\dot B}$ (the same is true for matrices $\Gamma^{\dot B}$ with matrix components $\Gamma^{\dot B}_{AI} := \Gamma^I_{A\dot B}$). Thus the matrices Γ^A provide the same algebraic structure as the matrices Γ^I and we can define the analogous antisymmetrized products $\Gamma^{ABCD\ldots}$ as in (B.4) with the same properties and analogous formulas as in (B.7), (B.8) and (B.9) will hold for them. In addition we can reinterpret different expressions in the tensor products (B.9). A particular example that was used in the main text is:

$$
\Gamma^{I}_{C\dot C}\Gamma^{IJ}_{AB} = \Gamma^{C}_{I\dot C}\Gamma^{AB}_{IJ} = -(\Gamma^{AB}\bar\Gamma^{C})_{J\dot C} = -(\Gamma^{ABC}_{J\dot C} + 2\,\Gamma^{[A}_{J\dot C}\delta^{B]C})\,,
\tag{B.10}
$$

with $\Gamma^{ABC}_{I\dot C} \equiv \Gamma^{IJ}_{[AB}\Gamma^{J}_{C]\dot C}$. In the main text we also use the fact that the adjoint representation of $so(8)$ can be written as

$$
\mathbf{28} = (\mathbf{8_v}\otimes\mathbf{8_v})_{\rm alt} = (\mathbf{8_s}\otimes\mathbf{8_s})_{\rm alt} = (\mathbf{8_c}\otimes\mathbf{8_c})_{\rm alt}\,,
\tag{B.11}
$$

which allows to label tensors in this representation by different antisymmetric index pairs, e.g.

$$
W_{IJ} \equiv \tfrac14\Gamma^{IJ}_{AB}W_{AB} \quad , \quad W_{AB} \equiv \tfrac14\Gamma^{IJ}_{AB}W_{IJ} \quad , \quad W_{\dot A\dot B} \equiv \tfrac14\bar\Gamma^{IJ}_{\dot A\dot B}W_{IJ}\,, \quad \text{etc.}\,.
\tag{B.12}
$$

C. $so(2,1)$ spinor conventions

All spinors appearing in the main text, superspace coordinates or fields, are Majorana spinors in $2+1$-dimensional space-time. Our metric convention is $\eta_{\mu\nu} = (-,+,+)$

[14] We do not introduce a new symbol for these matrices but take the index name from the range $A, B, C..$ as opposed to $I, J, K...$ as part of the defining symbol, in particular this means for example $\Gamma^{A=1} \neq \Gamma^{I=1}$.

and we choose a Majorana representation for the gamma-matrices[15]

$$\{\gamma^\mu, \gamma^\nu\}^\alpha{}_\beta = 2\eta^{\mu\nu}\delta^\alpha{}_\beta \quad . \tag{C.1}$$

Thus the matrices $\gamma^{\mu\ \alpha}{}_\beta$ are real and the Majorana condition on spinors imply that they are real two component spinors. Spinor indices are raised/lowered by the epsilon symbols with $\varepsilon^{12} = \varepsilon_{12} = 1$ and choosing NW-SE conventions

$$\varepsilon^{\alpha\gamma}\varepsilon_{\beta\gamma} = \delta^\alpha_\beta \ , \quad \lambda^\alpha := \varepsilon^{\alpha\beta}\lambda_\beta \Leftrightarrow \lambda_\beta = \lambda^\alpha\varepsilon_{\alpha\beta} \ . \tag{C.2}$$

Introducing the real symmetric matrices $\sigma^\mu_{\alpha\beta} := \gamma^{\mu\ \rho}{}_\beta\ \varepsilon_{\rho\alpha}$ and $\bar\sigma^{\mu\ \alpha\beta} := (\varepsilon \cdot \sigma^\mu \cdot \varepsilon)^{\alpha\beta} = -\varepsilon^{\beta\rho}\ \gamma^{\mu\ \alpha}{}_\rho$ a three vector in spinor notation writes as a symmetric real matrix as

$$v_{\alpha\beta} := \sigma^\mu_{\alpha\beta}\ v_\mu \quad \Rightarrow \quad v^\mu = \tfrac{1}{2}\bar\sigma^{\mu\ \alpha\beta}\ v_{\alpha\beta} \ . \tag{C.3}$$

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